Remarks on AriasMarco-Schüth's paper entitled: "Local symmetry of harmonic spaces as determined by the spectra of small geodesic spheres"

Zoltán Imre Szabó ***

Abstract

The main goal in this paper is to point out that quantity $||\nabla R||^2(p)$ on a harmonic space can not be determined by the spectra of local geodesic spheres or balls, therefore the main results of [AM-S] (quoted in the title) are wrong. My strong interest in this theorem is motivated by the fact that it contradicts some of my isospectrality examples constructed on geodesic spheres and balls of certain harmonic manifolds. The authors overlooked that the Lichnerowicz identity is not determined by the given spectral data, and so is the final crucial equation obtained by eliminating with the Lichnerowicz identity. In short, the above theorem has falsely been established by spectrally undetermined identities which can not be computed (determined) by the spectra of local geodesic spheres. More complicated spectrally undetermined functions cause the problems in case of local geodesic balls. I describe also a strong physical argument which clearly explains why the manifolds appearing in my examples are isospectral.

It must be pointed out, however, that a very remarkable new idea, namely, the asymptotic expansion of the heat invariants $a_k(p,r)$ defined on geodesic spheres, $S_p(r)$, is introduced in the paper. It can be used for developing both geometric uncertainty theory and global vs. local spectral investigations. Among my contributions to this developing field is the following statement: Average volumes, $\int vol(B_p(r))dp$ resp. $\int vol(S_p(r))dp$, of geodesic balls resp. spheres are generically not determined by the spectra of compact Riemann manifolds. This theorem interestingly contrasts the statement asserting that the volume of the whole manifold can be determined in terms of this global spectrum. In other words: Music written for local drums can not be played, in general, on the whole manifold.

*Partially supported by NSF grant DMS-0604861

**Lehman College of CUNY, Bronx, NY 10468,USA, and Rényi Institute, Budapest, POBox 127, 1364 Hungary.

1 A short history of harmonic manifolds.

Several equivalent definitions of harmonic manifolds are known in the literature. By one of them, they are the Riemannian manifolds yielding the local mean value theorem of harmonic functions. By an other characterization, they are the manifolds where the density function $\sqrt{\det(g_{ij})(q)} = \theta_p(q)/r^{n-1}$, defined on local normal coordinate systems about an arbitrary point $p \in M^n$, is radial, that is, it depends just on the distance $r(p,q)$. One can proof then that density function $\theta_p(r)$ does not depend on the reference point p either.

The latter characterization clearly shows that all locally two-point homogeneous manifolds are harmonic. The converse direction concerned, Lichnerowicz (1946) showed that harmonic manifolds satisfying $dim M^n = n \leq 4$ are exactly the locally 2-point homogeneous manifolds. The question if this statement is true also in higher dimensions, became known as Lichnerowicz conjecture concerning harmonic manifolds. In 1990, this problem was affirmatively answered on compact manifolds having finite fundamental groups, by this author in [Sz1]. Due to the fact that all harmonic manifolds are Einstein, they have constant scalar curvature, which is obviously strictly positive in the above compact cases. By Myers' and Cheeger-Gromoll's theorems, this affirmative answer extends also onto complete harmonic manifolds satisfying $Scal \geq 0$ [Sz2]. However, extension onto non-compact manifolds satisfying $Scal < 0$ defied every effort used by this author.

Then, in 1992, without knowing anything about the Lichnerowicz conjecture, Damek and Ricci [DR] accidentally discovered infinitely many non-compact and locally non-symmetric harmonic manifolds. Their statement says that the natural invariant metrics on the solvable extensions, $SH_l^{(a,b)}$, of Heisenberg-type groups, $H_l^{(a,b)}$, are harmonic. Such manifolds exist for any $l \geq 0$. Later we investigate families defined by fixed values $(a+b)$ and l. If $l \neq 3 \mod 4$, the metrics in such a family are isometric. However, if $l \neq 3 \mod 4$, two metrics in the same family are locally non-isometric, unless one of them is associated with (a,b) and the other with (b,a). Particularly interesting are the families defined for $l = 3$. In fact, the metric on $SH_3^{(a+b,0)} \simeq SH_3^{(0,a+b)}$ is symmetric while the others are locally non-symmetric.

Due to this discovery, the investigations were intensified both on compact and non-compact harmonic manifolds. Since the Damek-Ricci examples do not have compact factors, the question arises if the conjecture can be established on all compact manifolds and not just on those satisfying

$Scal \geq 0$. By using entropy theory of geodesic flows, this question has been positively answered on closed manifolds having negative sectional curvature, by G. Besson, G. Courtois and S. Gallot [BCG], in 1995. Since manifolds of strictly negative constant scalar curvature can have sectional curvature of changing sign, this question is still open on closed manifolds belonging to this category. In his very recent paper [Kn], Knieper confirms the Lichnerowicz conjecture for all compact harmonic manifolds without focal points or with Gromov hyperbolic fundamental groups. These results strongly suggests that the conjecture must be true on all manifolds that have compact Riemannian factors. The Damek-Ricci spaces have also been very intensely investigated in the past two decades. In this short list of achievements let it be mentioned just J. Heber's result [H], who, in 2006, proved that a homogeneous harmonic space is either a model space or a Damek-Ricci space. The existence of inhomogeneous harmonic manifolds is an other important open question in this field.

By this author, spectral investigations of harmonic manifolds were initiated. Since then, this field has been developed into several different directions. This paper can focus mostly on the investigations of this author. They inevitably involve also the Heisenberg type groups whose solvable extensions are the Damek-Ricci spaces. Several different isospectrality examples have been constructed. The isospectral domains arise always on the members of a family $H_l^{(a,b)}$ (resp. $SH_l^{(a,b)}$) defined by the same $(a+b)$ and l. The AM&S-paper refers to the examples constructed in [Sz4, Sz5], where the isospectrality is confirmed for any pair of geodesic balls resp. spheres having the same radius r. The AM&S-theorem contradicts the examples constructed on the particular family $SH_3^{(a,b)}$, where the metric on $SH_3^{(a+b,0)} \simeq SH_3^{(0,a+b)}$ is symmetric while the others in the isospectrality family are locally non-symmetric. What makes this situation more serious is that, in 2005, Hagen Fürstenau [F] discovered mathematical difficulties in the construction of the intertwining operator introduced in [Sz4, Sz5]. But, soon thereafter, this author did come up with the solution of this problem [Sz6, Sz8], which was publicly announced also at a conference held at CUNY, in February, 2006 [Sz7]. Since then, I found also a strong physical argument explaining why the isospectrality stated in [Sz4, Sz5] must be true. Namely, it turns out, that the Laplacian on the investigated manifolds can be identified with the Hamilton operators of elementary particle systems and the isospectrality is equivalent to the C-symmetry obeyed by these physical systems. This solution (together with explaining the difficulties in [Sz4, Sz5]) are described in the last section of this paper.

2 Technicalities on harmonic manifolds.

Along a unit speed geodesic $c(r)$, the Jacobian endomorphism field $A_{c(r)}(.)$ acting on the Jacobi subspace $V_{c(r)} \perp \dot{c}(r) = u(r)$ is defined by the equations

$$A''_{c(r)} + R_{\dot{c}(r)} A_{c(r)} = 0, \quad A_{c(0)} = 0, \quad A'_{c(0)} = id, \tag{1}$$

where $R_{\dot{c}(r)}(.) = R_u(.) := R(.,u)u$ is the Jacobian curvature along $c(r)$. Its normalized version is denoted by $\mathbf{A}_{c(r)} = \frac{1}{r} A_{c(r)}$. The power series of invariants $Tr\mathbf{A}^k_{c(r)}$ can directly be computed by these formulas. For instance, for $k = 1$ we have:

$$\begin{aligned} \mathbf{A}_{c_p(r)} = id + \frac{1}{6} r^2 R_{\dot{c}_p(0)} + \frac{1}{12} r^3 R'_{\dot{c}_p(0)} + \frac{r^4}{5!}(R^2_{\dot{c}_p(0)} + 3R''_{\dot{c}_p(0)}) \\ + \frac{r^5}{6!}(4R'_{\dot{c}_p(0)} R_{\dot{c}_p(0)} + 2R_{\dot{c}_p(0)} R'_{\dot{c}_p(0)} + 4R'''_{\dot{c}_p(0)}) + O(r^6), \end{aligned} \tag{2}$$

where $c_p(r)$ indicates that it starts out from $p = c_p(0)$ and $R'_{\dot{c}_p(0)}(.) = R'_{u_p}(.) = (\nabla_{u_p} R)(., u_p)u_p$, etc.

Among these invariants the density $\theta_{c(0)}(c(r)) = det\, A_{c(r)}$ plays the most important role in this paper. Its power expansion is usually computed by Legendre's recursion formulas. Before providing the corresponding formulas computed by this direct method, we have to point out that some of the basic objects introduced here are different from those appearing in [AM-S] or [Be]. As opposed to these texts, the Laplacian is a negative semi-definite operator and the curvature is defined by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ in all of my papers quoted later. In order to keep the exposition simple, in this section we keep the AriasMarco-Schüth-definition, $R(x,y)z = -\nabla_x \nabla_y z + \nabla_y \nabla_x z + \nabla_{[x,y]} z$, for the curvature, but will return back to our definition in the rest part of this paper. There is a difference also regarding θ in this paper which is associated with $det\, \mathbf{A}$ in [AM-S]. It should also be mentioned that some of the constants appear in [Be] incorrectly not just in those formulas which have been corrected in [AM-S] but also in some other expressions which are quoted in this paper (these errors and their effects on my computations were pointed out to me by D. Schüth). In terms of these quantities we have:

Theorem 2.1. *(see [Be], Chapter 6). If M is harmonic then there exist constants $C, H, L \in \mathbb{R}$ such that for all $p \in M$ and all $u \in T_pM$ with $|u| = 1$*

we have:

$$Tr(R_u) = C; \quad \text{in particular :} \tag{3}$$

$$Tr(R_u^{(k)}) = 0, \quad \text{for all derivatives of order } k \in \mathbb{N}. \tag{4}$$

$$Tr(R_u R_u) = H; \quad \text{in particular :} \tag{5}$$

$$Tr(R_u R_u') = 0 \quad \text{and} \tag{6}$$

$$Tr(R_u R_u'') = -Tr(R_u' R_u'). \tag{7}$$

$$Tr(32 R_u R_u R_u - 9 R_u' R_u') = L. \tag{8}$$

The first five conditions imply that harmonic manifolds are Einstein, furthermore, the curvature has constant norm $||R||^2 = R_{abcd}R^{abcd}$. Due to the Einstein property, these manifolds are analytic and the unique radial density $\theta(r)$ is uniquely determined by the constants appearing in the Taylor expansion. Such a density function does not determine the metric. See, for instance, that the above families of Damek-Ricci spaces share the very same density function $\theta(r)$. An other surprising feature of these examples is that in families $SH_3^{(a,b)}$ member $SH_3^{(a+b,0)}$ is symmetric while the others are locally non-symmetric.

In this paper particular attention is paid to condition (8). It should be emphasized that this condition declares this curvature-expression to be constant just for this combination but it says nothing about the terms $R_u R_u R_u$ and $R_u' R_u'$ which can very well be non-constant functions even along a fixed geodesic. The same density function can manifest itself very differently. For instance, on $SH_3^{(a+b,0)}$, constant L is incorporated completely into $R_u R_u R_u$ but on the other members the same L is distributed both into $R_u R_u R_u$ and $R_u' R_u'$. Even the proportion of the two terms depends on the members of a family.

By integrating these constants on the unit sphere formed by unit vectors u in the tangent space $T_p(M)$ one eliminates direction u and obtains the curvature conditions depending just on $p \in M$. This computations result the following well known formulas:

$$||R||^2 = \frac{2n}{3}((n+2)H - C^2), \tag{9}$$

$$32(nC^3 + \frac{9}{2}C||R||^2 + \frac{7}{2}\hat{R} - \mathring{R}) - 27||\nabla R||^2 = n(n^2+6n+8)L, \tag{10}$$

where $\hat{R}$ and $\mathring{R}$ are defined in an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space by the formulas:

$$\hat{R} = \sum_{i,j,p,q,r,s} R_{ijpq}R_{pqrs}R_{rsij}, \quad \mathring{R} = \sum_{i,j,p,q,r,s} R_{ipjq}R_{prqs}R_{risj}. \tag{11}$$

Once again, the harmonicity conditions impose just this single equation onto a tensor field having a large number of components. Thus, beyond this relation, it provides no further information about the main players; $\hat{R}$, $\mathring{R}$, and $||\nabla R||^2$; in this formula. In other words, these individual functions are out of touch of the harmonicity conditions. Due to Lichnerowicz, there is an other independent equation:

$$-\frac{1}{2}\Delta||R||^2 = 2C||R||^2 - \hat{R} - 4\mathring{R} + ||\nabla R||^2 \tag{12}$$

which holds true not just on harmonic but on all Einstein manifolds. On harmonic manifolds, relation $||R||^2 = constant$ implies (see [Be], Proposition 6.68):

$$112\hat{R} - 32\mathring{R} = 27||\nabla R||^2 + Q_1, \quad \hat{R} + 4\mathring{R} = ||\nabla R||^2 + Q_2, \tag{13}$$

$$\text{therefore,} \quad \hat{R} = \frac{7}{24}||\nabla R||^2 + Q_3, \quad \mathring{R} = \frac{17}{96}||\nabla R||^2 + Q_4, \tag{14}$$

where Q_1, Q_2, Q_3, Q_4 are constants which can be expressed in terms of C, H, and L. (The second formula on line (13) appears with an incorrect factor 2 (before $||\nabla R||^2$) in [Be].)

It should be pointed out that there is a substantial difference between the equations (10) and (12). The first one is defined by integrating the constant function (8) on the unit sphere while no such natural integral representation exist for the Lichnerowicz identity. This identity is originally established by operations defined just on M which are nothing to do with curvature quantities such as $R'_u R'_u$;...; e. t. c., defined in terms of directions u. This detachment from functions defined by radial expansions strongly indicates that the Lichnerowicz identity may cause serious difficulties in investigating spectrally determined identities obtained by the radial expansion of the heat invariants $a_k(p, r)$. In fact, it will turn out that this identity is not encoded into the spectra of local geodesic spheres, by any means, and its application for reformulating spectrally determined identities produces spectrally undetermined ones. This is why the identity established in [AM-S] in the final step is not spectrally determined. Therefore, it can not prove the desired spectral determinacy stated in the paper. There is also an other explanation for this problem which does not use Lichnerowicz identity at all. Although it is defensible in case of geodesic spheres, it develops into a major argument against the AM&S-statement established by the spectra of local geodesic balls. The precise details are as follows.

3 The AriasMarco-Schüth local heat invariants.

The first few heat invariants of a compact manifold S are given by:

$$a_0(S) = vol(S),\ a_1(S) = \frac{1}{6}\int_S scal\, dvol_S, \tag{15}$$

$$a_2(S) = \frac{1}{360}\int_S (5scal^2 - 2||Ric||^2 + 2||R||^2)\, dvol_S. \tag{16}$$

They are defined as coefficients in the asymptotic expansion

$$Tr(exp(t\Delta)) = \sum_{i=0}^{\infty} exp(\lambda_i t) \sim (4\pi t)^{-dim(S)/2} \sum_{k=0}^{\infty} a_k(S) t^k \tag{17}$$

where $t \downarrow 0$ and $\lambda_i \to -\infty$ denote the eigenvalues of Laplacian Δ with multiplicities.

Arias-Marco and Schüth furnished the spectral data of local geodesic spheres by the asymptotic expansions of radial functions $a_k(S_p(r))$, where $S_p(r)$ denotes the geodesic sphere of radius r about p. The first few terms are explicitly computed for $a_2(S_p(r))$. The following theorem is a combination of Propositions 3.2 and 4.2 of [AM-S], however, it is not a literal transcription of those statements. In order to prepare the arguments discussed below, this transcription focuses also on the eliminating steps (24)-(27). These are just briefly explained in [AM-S] without introducing any formulas.

Theorem 3.1. *[AM-S] Consider an n-dimensional harmonic space M with constants $C; H; L$, introduced above. Let $p \in M$ and let u be a unit vector in $T_p(M)$. Then*

$$||Ric^{S_p(r)}||^2_{exp(ru)} = \alpha_{-4} r^{-4} + \alpha_{-2} r^{-2} + \alpha_0 + \alpha_2 r^2 + O(r^3), \tag{18}$$

$$\text{and} \quad ||R^{S_p(r)}||^2_{exp(ru)} = \beta_{-4} r^{-4} + \beta_{-2} r^{-2} + \beta_0 + \beta_2 r^2 + O(r^3) \tag{19}$$

for $r \downarrow 0$, where the coefficients α_i and β_i for $i = -4, -2, 2$ are constants depending only on n, C and H. Moreover,

$$\alpha_2(u) = \hat{\alpha}_2 + \frac{1}{16} Tr(R'_u R'_u) \tag{20}$$

$$\text{and} \quad \beta_2(u) = \hat{\beta}_2 + \frac{4}{9}\sum_{i=1}^{n} Tr(R_u \circ R(e_i, .) R_u e_i), \tag{21}$$

where the coefficients $\hat{\alpha}_2$ and $\hat{\beta}_2$ are constants depending only on $n; C; H; L$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_p(M)$.

By integrating on the unit sphere $S_1(0_p) \subset T_p(M)$, whose volume is denoted by ω_{n-1}, we get:

$$\int_{S_1(0_p)} Tr(R'_u R'_u)du = \frac{3\omega_{n-1}}{n(n+2)(n+4)}||\nabla R||^2(p), \quad (22)$$

$$\int_{S_1(0_p)} \sum_{i=1}^{n} Tr(R_u \circ R(e_i,.)R_u e_i)du = (nC^3 + 2\mathring{R}(p) - \frac{1}{4}\hat{R}(p))\frac{\omega_{n-1}}{n(n+2)}. \quad (23)$$

By using (14) for eliminating terms $\mathring{R}(p)$ and $\hat{R}(p)$) from (23) we get:

$$2\mathring{R}(p) - \frac{1}{4}\hat{R}(p) = \frac{9}{32}||\nabla R||^2(p) + Q(n, C, H, L), \quad (24)$$

where Q is a constant depending only on constants appearing in its argument, moreover:

$$\overline{\alpha}_2 := \frac{1}{\omega_{n-1}} \int_{S_1(0_p)} \alpha_2(u)du = \tilde{\alpha}_2 + \frac{3}{16n(n+2)(n+4)}||\nabla R||^2(p), \quad (25)$$

$$\overline{\beta}_2 := \frac{1}{\omega_{n-1}} \int_{S_1(0_p)} \beta_2(u)du = \tilde{\beta}_2 + \frac{9}{32n(n+2)}||\nabla R||^2(p), \quad (26)$$

where $\tilde{\alpha}_2$ and $\tilde{\beta}_2$ are constants depending only on n, C, H, and L. Thus the scrutinized coefficient is reborn-ed in the new form:

$$\tilde{\beta}_2 - \tilde{\alpha}_2 + \frac{9n - 30}{32n(n+2)(n+4)}||\nabla R||^2(p). \quad (27)$$

The computation proceeds by inserting this asymptotic expansion of $a_2(S_p(r))$ into the integral formula $\int_{S_p(r)} a_2(S_p(r))\frac{dvol(S_p(r))}{vol(S_p(r))}$, where the integration is taken with respect to the normalized density $\frac{dvol(S_p(r))}{vol(S_p(r))}$. It is particularly important at this step that the density is constant regarding r, that is, it appears as the normalized density on the Euclidean unit sphere. Therefore, in the asymptotic expansion of the whole integral formula this density has no effect on the expansion. Also note that, due to the uniquely determined density function $\theta(r)$ on a fixed harmonic manifold (or for a whole family $SH_l^{(a,b)}$), the constants like C, H, L,...e. t. c. are spectrally determined values. After term by term integration of the corresponding coefficient of the asymptotic expansion, the process is finished by eliminating $\mathring{R}(p)$ and $\hat{R}(p)$ by formulas (14). Then the final conclusion is this: Since the coefficients of the asymptotic expansion as well as the constant values are spectrally determined, therefore also $||\nabla R||^2(p)$ must be spectrally determined. This statement is considered also regarding the spectra of local geodesic balls. This case is discussed in the next section.

4 Examining the AM&S-proof.

4.1 Case of geodesic spheres.

It must be emphasized again that the concerns brought up against this proof appear in the final eliminating step. All preceding, highly non-trivial computations are performed with impressive carefulness. The key problem has already been indicated at the introduction of the Lichnerowicz identity. According to those comments, this equation does not have any natural integral representation and its application cuts the computations completely off from the spectral data. It seems to be that it is not encoded into the spectra of local geodesic spheres at all, thus, by adding it to a local heat invariant, it may change it into a quantity which is not local heat invariant any more. Next, we show that exactly this is the case. We will prove that this identity is really a spectrally undetermined object and when it is added, in the form of the above described linear combination with (10), to the local heat invariant computed in the above theorem, then this elimination process changes this spectrally determined invariant to a spectrally undetermined curvature expression which can not be produced as linear combination of "true" local heat invariants obtained by direct radial expansions of several heat invariants $a_k(p,r)$. The conclusion with such a "fake" local heat invariant gives, of course, a false proof for the desired spectral determinacy of $||\nabla R||$.

In order to establish these statements, one should define the spectral data at a fixed point $p \in M$. It is defined by the linear space of curvature expressions obtained by radial expansions of heat invariants $a_k(p,r)$. They can be computed by integrating the coefficients defined by the radial expansions of the heat invariant functions regarding du_p. The evaluations of these integrals provide the curvature expressions which can be written up as identities such that the right sides of these identities are thought to be the integral values which are equal to the curvature expressions obtained by the AriasMarco-Schüth expansions. On the left sides, the different curvature symbols can be considered as independent objects (unknowns). These curvature terms are defined on M, that is they depend on p and not on the directions u_p. This identity interpretation of these curvature expressions better indicates where these "true" local heat invariants come from. Thus the given spectral data at p is a linear space spanned by the identities obtained by the AriasMarco-Schüth radial expansions of the complete set of heat invariants.

In order to define the complete linear space of local heat invariants at p, one should introduce a naturally defined pre-Hilbert norm on this linear

space. It is defined by integrating the products $F_1(u)F_2(u)$ of infinitesimal heat invariant functions by means of du. By the topological closure associated with this pre-Hilbert norm, one can define the *spectral-identity-space* $SIS(p)$, which Hilbert space contains all spectrally determined identities attached to the given spectral data. The identities belonging to $SIS(p)$ are the only ones which are determined by the given spectral data. Also note that this spectral data determines just the identities obtained by integrating the infinitesimal heat invariant functions, but, it does not determines the pre-functions $F(u)$ from which these identities are derived.

Because of this new interpretation, the identities introduced so far must be rewritten in new appropriate forms. This reformulation mostly concerns the constants appearing on the left sides of these identities. Also these terms must appear as combinations of curvature terms (unknowns) obtained by integrating the pre-functions. The values standing on the right sides are just formally symbolized by the integral $\int F(u)du_n$, where du_n denotes normalized measure, whose actual evaluation provides the curvature terms, depending just on p, on the left sides of these equations. To this end, consider the constants $C(u), H(u)$, and $L(u)$ defined in formulas (3)-(8) as functions of u which are actually defined by the left sides of those equations. Then L on the right side of (10) must be substituted by $\int L(u)du_n$ while C^3 and $C||R||^2$ on the left side should be replaced by the curvature expressions obtained by actual evaluations of integrals $\int C^3(u)du_n$ and $\frac{2n}{3}\int C(u)((n+2)H(u)-C^2(u))du_n$. These integrals provide linear combinations of terms such as $Scal_M^3(p)$ and $Scal_M(p)||R||^2(p)$. On non-Einstein manifolds these expressions are more complicated involving also the Ricci tensor and some other curvature terms. Similar reformulation should be implemented in (12), (20), (21), (23)-(26) and also radial function $scal^2$ in (16) should be expanded.

It is important to understand that this reformulation is not just a kind of fussiness. For instance, identities obtained by evaluating integrals like $\int C^3(u)du_n$ and $\frac{2n}{3}\int C(u)((n+2)H(u)-C^2(u))du_n$ are not encoded into the spectra of local geodesic spheres, meaning that they can not be obtained from the heat invariants $a_k(p,r)$ by radial expansions and linear combinations. That is, these curvature identities are not individually encoded into $SIS(p)$. The explanation for this phenomenon is that they can be derived not from the original pre-functions defining the identities in $SIS(p)$ but from functions defined by powering or multiplying the original heat invariant pre-functions. They usually define curvature identities lying in the complement of $SIS(p)$. In the following discussions such equations are called *powered curvature identities*.

Although identities associated with $\int C^3(u)du_n$ or $\int C(u)H(u)du_n$ do not show up in $SIS(p)$, yet, on harmonic manifolds, they are determined by the spectra of local geodesic spheres. This statement is true for any identity defined by $\int T_1(u)T_2(u)\dots T_k(u)du_n$, where functions $T_i(u)$ are density-Taylor coefficients obtained by expanding θ regarding directions u. Indeed, on harmonic manifolds, the Taylor coefficients of $\int \theta_p^k(u,r)du_n = (\int \theta_p(u,r)du_n)^k$ define spectrally determined identities, for any power $k \in \mathbb{N}$, which are associated with integrals $\int T_1(u)T_2(u)\dots T_k(u)du_n$. In order to set up a spectral data which contains also these powered density curvature identities, system $\{a_k(p,r)\}$ should be extended by the functions $\int \theta_p^k(u,r)du_n$ and extended spectral identity space, $SIS^{ex}(p)$, should be defined by this complete extended function-system. This space incorporates all identities which are determined by the spectra of local geodesic spheres of harmonic manifolds.

It is interesting to see that how do these powered density functions define the powered curvature terms. We demonstrate this by considering the Taylor coefficient containing L in the expansion of θ. It can be proved that, by increasing power k by 1, the contribution to the previous function is a linear combination of terms C^3, CH, and L. Thus the powered identities associated with these pre-functions can be individually defined by powered density functions θ, θ^2, and θ^3. The AM&S paper uses normalized density for the expansions. In this paper we consider also the natural expansion $a_k(p,r)$ without normalizing the density θ in the integral formula. By the above discussion, this exchange of normalized density to the non-normalized one defines new coefficients for the terms L, C^3, and CH in the investigated Taylor coefficient associated with L.

After these definitions we are able to explain why does the AM&S-proof breaks down after eliminating with the Lichnerowicz identity. The ultimate problem is that the curvature expression standing on the left side of this equation does not show up among the spectrally determined curvature expressions. That is, this identity is not encoded neither in $SIS(p)$ nor in $SIS^{ex}(p)$. This statement can be seen by proving that it is independent from both identity spaces.

This observation is demonstrated, first, for the case satisfying $||\nabla R||(p) \neq 0$. Since it is independent from the higher order equations expressed in terms of higher order curvature terms, it is enough to see that it is linearly independent from the lower-order spectrally determined identities introduced so-far. In case of $SIS(p)$, the problem reduces to the independence of (12) from the system consisting (10) and the identity resulted by computations

(20)-(23). The desired independence can immediately be decided by considering only the main curvature expressions determined by terms $\hat{R}$, $\mathring{R}$, and $||\nabla R||^2$. In case of $SIS^{ex}(p)$, the data contains also other identities, but all new identities have main curvature expressions proportional to the main term of (10). Thus the independence is established also in this case.

This independence from the whole $SIS^{ex}(p)$ means that the curvature expression on the left side of the Lichnerowicz identity can not be produced by linear combinations of curvature expressions obtained from the expansions of $a_k(p,r)$ and $\theta^k(p,r)$. It is obvious, that, the addition of a non-trivial constant-times of the Lichnerowicz identity to an identity belonging to $SIS^{ex}(p)$ changes it to one which does not belong $SIS^{ex}(p)$ any more. But exactly this happens during the considered elimination which can be described such that a certain linear combination of the Lichnerowicz identity and (10) is added to the spectrally determined identity resulted in (20)-(23). Since (10) is in $SIS^{ex}(p)$, it still keeps the identity subjected to elimination in $SIS^{ex}(p)$. But the Lichnerowicz identity results an identity which is certainly not in $SIS^{ex}(p)$ any more. Therefore, this curvature identity (expressed in terms of $\int L(u)du_n, \int C^3(u)du_n, \int CH(u)du_n$, and $||\nabla R||^2$ but not containing neither $\hat{R}$ nor $\mathring{R}$) can not be obtained by the radial expansions of functions $a_k(p,r)$ and $\theta^k(p,r)$. Since all terms but $||\nabla R||^2$ can be obtained by such expansions, thus $||\nabla R||^2$ is the only term which can not be computed by such expansions. This proves that it is not a spectrally determined quantity.

In case of $||\nabla R||^2 = 0$, the above investigated identities have only two main terms, $\hat{R}$ and $\mathring{R}$, thus the independence of the Lichnerowicz identity from the other two identity is not insured by the above arguments. If it is independent, then one can imply in the same way that the elimination produces a spectrally undetermined identity. But such an identity can not say anything about the spectral determinacy of local symmetry. If it is in $SIS^{ex}(p)$, then terms $\hat{R}$ and $\mathring{R}$ can be eliminated from the investigated identity. After this operation only terms associated with $\int C^3(u)du_n, \int C(u)H(u)du_n$ remain there. The only information provided by this step is that they are spectrally determined quantities. But this is a trivial information which does not imply the spectral determinacy of local symmetry, either.

4.2 Rudimentary vs. refined $SIS(p)$; Proper eliminations.

Before examining the proof of the AM&S-statement in case of geodesic balls, we point out some of the rudimentary features of $SIS(p)$. As it turnes out,

they can easily lead to misinterpretation of the meaning of the spectral determinacy of the curvature expressions defined by the given spectral data. Although the problems caused by these features can be fixed in case of geodesic spheres, but they are not so in case of geodesic balls.

The primary problem about $SIS(p)$ is that the curvature expressions obtained by evaluating integrals $\int F_c(u)du_n$, where $F_c(u)$ is an c^{th}-order expansion coefficient defined by radial expansion of a heat invariant $a_k(p,r)$, are considered without any reference to the degree c. It is obvious that this data can not be used for constructing functions depending on r. Such functions can be defined just by the complete expressions $\int F_c(u)du_n r^c$ involving r^c. This missing factor turnes the above elimination process to a very ambiguous operation. Indeed, if a term from $\int F_c(u)du_n$ can be eliminated by a linear combination $\sum_i Q_i \int F_{c_i}(u)du_n$ where none of the degrees c_i is equal to c, then that term is still present in $\int F_c(u)du_n r^c - \sum_i Q_i \int F_{c_i}(u)du_n r^{c_i}$. Furthermore, no linear combination $A(p,r) = \sum_i P_i a_{k_i}(p,r)$ exists whose coefficient of degree c is equal to the curvature expression resulting by the considered elimination.

In order to explain this situation by a concrete example, suppose that density coefficient L appears in the coefficient $(\dots, QL, \dots)_{n+1} r^{n+1}$ of $(n+1)$-degree in the natural expansion of a heat invariant $a_k(p,r)$ what we wish to eliminate by the L appearing in the coefficient of $(n+5)$-degree in the natural expansion of the density heat invariant $\int \theta(p,r)du_n$. Then the coefficient $(\dots, 0, \dots)_{n+1}$ resulting by elimination can be decomposed as $(\dots, 0, \dots)_{n+1} = (\dots, QL, \cdots - QL)_{n+1}$, meaning that it has $a_k(p,r)$- and $\int r^{-4}\theta(p,r)du_n$-components. Since the latter one is not a heat invariant anymore, this elimination process involves new functions which are not listed among the elements of the spectral data and produces functions like $L(u)(r^{n+1} - r^{n+5})$ which still contain L and which are not consistent with the requirement of radial constructibility of these functions. It is clear that all these problems are caused by eliminating terms appearing in coefficients of distinct radial degrees with each other. These arguments can not be brought up for eliminations performed by coefficients having equal degrees. Thus the only proper elimination in this situation is the elimination of QL_{n+1} appearing in $(\dots, QL, \dots)_{n+1}$ by the L_{n+1} appearing in the corresponding coefficient defined by the natural expansion of $\int r^{-4}\theta(p,r)du_n = r^{-4}\int \theta(p,r)$. Note that these coefficients are still spectrally determined by the spectra of local geodesic spheres, because, modulo certain multiplicative constants, just the degree defined by $\int \theta(p,r)du_n$ must be lowered by 4 in order to get the corresponding degree regarding $\int r^{-4}\theta(p,r)du_n$. Thus the proper treatment of these problems requires just the extension of the original spectral

data $\{a_k(p,r)\}$ to the data $\{a_k(p,r), \int (r^{-4}\theta^k(p,r)du_n\}$.

A precise establishment of this refined data, $FineSIS(p)$, is as follows. In the first step the same type of expansion should be chosen regarding each heat invariant $a_k(p,r)$, which can be either the natural expansion or the normalized density expansion. The degree, c, associated with a spectral identity $\int F_c(u)du_n = (Q\hat{R} + \dots)_c$ is indicated in the lower right corners. They are the coefficients of r^c's appearing in the complete radial expansion formulas. For a fixed c, the $SIS_c(p)$ is generated by all heat invariant coefficients of degree c. When this generator system is extended by coefficients, of degree c, defined for all functions $\int (r^{2d}\theta^k(p,r))du_n\}$, then the corresponding extended coefficient space of degree c is denoted by $SIS_c^{EXT}(p)$. In order to avoid operations among coefficients of different degrees, the total refined spectral identity spaces must be defined by the direct sums $FineSIS(p) = \oplus_c SIS_c(p)$ resp. $FineSIS^{EXT}(p) = \oplus_c SIS_c^{EXT}(p)$. It is obvious that these spaces can be defined just for coefficients obtained from the expansions of $a_k(p,r)$ resp. $\int (r^{2d}\theta^k(p,r))du_n\}$.

These refined spectral identity spaces put an end to the greatest imperfection of the rudimentary $SIS(p)$'s which do not give any indication about the radial functions from which the elements of the rudimentary $SIS(p)$ are derived. In case of $FineSIS(p)$ and $FineSIS^{EXT}(p)$, however, any element $\sum_i Q_i \int F_c^{(i)}(u)du_n$ of $SIS_c(p)$ or $SIS_c^{EXT}(p)$ determines the expansion term $F_c(r) = \sum_i Q_i \int F_c^{(i)}(u)du_n r^c$ and there exist a function $F(r) = \sum_c F_c(r)$ in the space spanned by functions $\{a_k(p,r)\}$ resp. $\{a_k(p,r), \int r^{2d}\theta^k(p,r))du_n\}$ whose c^{th} expansion term is $F_c(r)$. Also the elimination process appears in a well defined clear form. If the above linear combination $\sum_i Q_i \int F_c^{(i)}(u)du_n$ eliminates a term then $F(r) = \sum_c F_c(r)$ is such a globally defined function whose c^{th} expansion coefficient is the curvature expression resulting by the elimination. Eliminations which can be described by such globally defined functions are called *proper eliminations*. Only such eliminations are acceptable to decide whether a term is spectrally determined by the given data. (It is obvious that all those curvature expressions are spectrally undetermined which can not be traced back to the functions $\{a_k(p,r), \int r^{2d}\theta^k(p,r)du_n\}$ in this way.) Eliminations performed by the elements of $SIS^{ex}(p)$ are called *rudimentary eliminations.* Let it be pointed out again that, by the above arguments, these latter operations do not carry out the desired elimination, and, to the curvature expression, which are the results of formal rudimentary elimination, no globally defined function exists which satisfies the above properties.

4.3 Case of geodesic balls.

A review of AM&S-computations. In [AM-S], the geodesic balls are considered regarding both the Dirichlet and Neumann conditions. These computations start out from the Branson-Gilkey formulas establishing the asymptotics of the Laplacian on a manifold with boundary. After adopting these generic formulas to harmonic manifolds, the Dirichlet coefficient $a_2^D(B)$ appears in terms of the shape operator $\sigma_u(r) = (\nabla\nu)_{|T_{exp(ru)}M}$ (where ν is the normal vector field on ∂B) in the following form:

$$a_2^D(B) = \frac{1}{360}\Big(\int_B P_1^D(C^2, H)dvol_B + \int_{\partial B}(P_2^D(..) + P_3(..))dvol_{\partial B}\Big), \quad (28)$$

$$\text{where} \quad P_1^D(C^2, H) = 5(nC)^2 - 2nC^2 + \frac{4}{3}n((n+2)H - C^2), \quad (29)$$

$$P_2^D(..) = 20nCTr(\sigma) - 8CTr(\sigma) + 16Tr(R_\nu \circ \sigma), \quad (30)$$

$$P_3^D(..) = \frac{40}{21}(Tr(\sigma))^3 - \frac{88}{7}Tr(\sigma)Tr(\sigma^2) + \frac{320}{21}Tr(\sigma^3). \quad (31)$$

The radial expansions of these functions are established by means of the following formulas:

$$Tr(\sigma) = (n-1)\frac{1}{r} - \frac{1}{3}Cr - \frac{1}{45}Hr^3 - \frac{1}{15120}Lr^5 + O(r^7), \quad (32)$$

$$Tr(\sigma^2) = (n-1)\frac{1}{r^2} - \frac{2}{3}C - \frac{1}{15}Hr^2 + \frac{1}{3024}Lr^4 + O(r^6), \quad (33)$$

$$Tr(R_\nu \circ \sigma) = \cdots + (-\frac{1}{1440}L + \frac{1}{96}Tr(R'_uR'_u))r^3 + \ldots, \quad (34)$$

$$Tr(\sigma^3) = \cdots + (\frac{1}{30240}L - \frac{1}{96}Tr(R'_uR'_u))r^3 + \ldots. \quad (35)$$

In the last two formulas only the r^3-coefficients of the corresponding series are indicated. Regarding the complete functions P_2^D and P_3^D, these coefficients can be written in the form

$$[P_2^D]_3(CH, L, Tr(R'_uR'_u)),\ [P_3^D]_3(C^3, CH, L, Tr(R'_uR'_u)), \quad (36)$$

where also the arguments (on which these coefficients depend) are indicated.

Similar formulas are established also for $a_2^N(B)$. Although $P_1^D(C^2, H) = P^N(C^2, H)$ and $P_2^D = P_2^N$, the third one:

$$P_3^N(..) = \frac{40}{3}(Tr(\sigma))^3 + 8Tr(\sigma)Tr(\sigma^2) + \frac{32}{3}Tr(\sigma^3). \quad (37)$$

is linearly independent from P_3^D. Thus, also $[P_3^N]_3(C^3, CH, L, Tr(R'_u R'_u))$ is a new linear combination of the arguments.

Normalized-density expansions and powered terms on balls. For the radial expansion of this local heat invariant, normalized density is used also in this part of the paper. It is computed just regarding the term defined on the boundary ∂B, in which case it means the expansion of

$$r \to \frac{1}{vol(S_r(p))} \int (P_2^D(..) + P_3(..)) dvol_{S_r(p)}\big). \tag{38}$$

In these discussions we need also the corresponding expansion of the volume function. By the previous formula, this means the expansion of

$$r \to \frac{vol(B_r(p))}{vol(S_r(p))} = \frac{1}{vol(S_r(p))} \int_0^r \int_{S_1(p)} \theta(u_p, \rho) du_p d\rho. \tag{39}$$

Note that this normalization concerns just the density defined at the endpoint $\rho = r$ of radial integration and it does not mean normalization regarding any ρ between 0 and r. For explicit computations of the higher order powered terms appearing in the formulas one must consider also the so called higher order k-volumes defined by $Vol_k(B_p(r)) = \int_{B_p(r)} \Theta^k(p, \rho) dvol_{B_p(r)}$, where $\Theta(r) = \det \mathbf{A}(c_p(r)) = A_0 + A_2 r^2 + A_4 r^4 + A_6 r^6 + \dots$. The exact formulas, regarding both the natural and normalized-density expansions of these functions, are described later in this section.

Before going into the details let it be mentioned in advance that in case of geodesic spheres the spectrally determined functions $Vol_k(S_p(r))$ had to be extra added to the set $\{a_k(p, r)\}$ of local heat invariants, because, they do not belong to them originally, but, on harmonic manifolds, their spectral determinacy can be established by formulas such as $\int \theta^b(u_p, r) du_n = (\int \theta(u_p, r) du_n)^b$ and $\int \Theta^b(u_p, r) du_n = (\int \Theta(u_p, r) du_n)^b$. This spectral determinacy can not be established for functions $Vol_k(B_p(r))$, however. Indeed, for $k > 0$, these functions are nothing to do with the heat invariants of local geodesic balls, furthermore, there does not exist a function-relation $Vol_k(B_p(r)) = F_k(vol(B_r(p))$ expressing the k-volume $Vol_k(B_p(r))$ in terms of the standard volume $vol(B_p(r)) = Vol_0(B_p(r))$, for all r. Thus neither the k-volumes nor the proper powered terms (obtained by using higher order k-volume-functions for eliminations) are determined by the spectra of local geodesic balls. Therefore this spectral data can not be extended by these spectrally undetermined functions.

Rudimentary vs. proper eliminations on balls. After this detour we turn back to the AM&S-proof, where the desired spectral determinacy of

$||\nabla R||^2$ is concluded just by a briefly explained argument. According to this explanation, just term L appearing in the above r^3-coefficients has to be eliminated by the L appearing in the corresponding coefficient of the radial expansion of the first heat invariant $a_0^D(p,r) = vol(B_r(p)) = a_0^N(p,r)$. (The elimination of this L is really necessary because it involves $Tr(\nabla_u R_u \nabla_u R_u)$.) However, a deeper insight into the details shows that this briefly explained step covers up numerous problems which discredit the AM&S-proof also in this case. One of them is indicated above. Even if this elimination were appropriate, there are still higher order spectrally undetermined powered terms left behind because of which term $||\nabla R||^2$ can not be spectrally determined. But in reality there are much more deeper reasons why this quantity is spectrally undetermined.

First of all, the computations below show that the L appears with different degrees regarding the two functions $a_0(p,r)$ and $a_2(p,r)$, with respect to both type of expansions. The degree computed by $a_2(p,r)$ is always lower by 2 than the degree computed by $a_0(p,r)$. This means that the proposed elimination is a typical rudimentary operation which leads to several contradictory situations. Remember that, in order to keep connection with the radial functions from which these coefficients are defined, this operation must be well defined in terms of these functions. Since radial functions $Q_1 L r^c$ and $Q_2 L r^{c+4}$ can not cancel out each other, the desired elimination of L does not take place, furthermore, the coefficient defined by the formal rudimentary elimination does not appear as a coefficient of a spectrally determined function, which, by definition, must be a linear combination of functions $\{a_k(p,r)\}$. In the top of these problems, also spectrally undetermined higher order powered terms are present in both functions. Since they can not cancel out each other, the proposed rudimentary operation is totally inappropriate for the desired elimination.

Proper eliminations can be implemented by functions

$$\int_{B_p(r)} \rho^{-4} \Theta^k(p,\rho) dvol_{B_p(r)}, \quad \text{where} \quad k \geq 0, \tag{40}$$

but this adjustment changes even $vol(B_p(r))$ into a spectrally undetermined function and it further worsens the spectral indeterminacy of the k-volumes of balls. Remember that, in case of geodesic spheres, the spectral determinacy of $\int r^s \theta^k(u_p,r) du_n = r^s \int (\theta(u_p,r))^k du_n$ is due to the indicated equation. Since ρ^{-4} is inside of an integration, such reconnection to the volume does not exist in case of geodesic balls. (By using multiple partial integrations, term ρ^{n-5} can be retraced to ρ^{n-1}. The complicated other terms produced by this process show then the impossibility of the reconnection of

these adjusted functions to the volume.) In other words, all these functions are spectrally undetermined, together with all those terms which can be eliminated by these functions in a proper elimination process. This means that the L, which can not properly be eliminated from $a_2(p,r)$ by a spectrally determined function, but just by a spectrally undetermined one, is spectrally determined in $a_0(p,r)$ whereas it is spectrally undetermined in $a_2(p,r)$. That is, the spectral determinacy of a term such as L is not an absolute but a relative concept. This property depends on the degree of the expansion term in which it shows up. This paradoxical statement points to the deep sitting ambiguity present in rudimentary spectral data defined for geodesic balls. It clearly explains also the error made in this part of [AM-S]. By the proposed elimination they tried to cancel out a spectrally undetermined quantity by a spectrally determined one. As it is explained above, this is a totally inappropriate operation from several point of view.

Explicit expansion formulas. In this section those expansions are explicitly computed which are mentioned in the above discussions. Expansion of $vol(B_r(p))/vol(S_r(p))$ appears in the form:

$$\frac{vol(B_r(p))}{vol(S_r(p))} = \frac{vol(B_r(p))}{vol'(B_r(p))} = r(V_0 + V_2 r^2 + V_4 r^4 + V_6 r^6 + \dots). \qquad (41)$$

The details below show that this expansion does not provide L in a pure form. Instead, it is accompanied with higher order powered terms. In fact, a simple calculation shows that coefficients V_k can be determined by the coefficients of expansions

$$\Theta(r) = \det \mathbf{A}(c_p(r)) = A_0 + A_2 r^2 + A_4 r^4 + A_6 r^6 + \dots, \quad (42)$$

$$vol(B_r(p)) = r^n(\frac{1}{n}A_0 + \frac{1}{n+2}A_2 r^2 + \frac{1}{n+4}A_4 r^4 + \frac{1}{n+6}A_6 r^6 + \dots) \quad (43)$$

by the recursive formula

$$V_k = \frac{1}{A_0}(\frac{1}{n+k}A_k - A_2 V_{k-2} - A_4 V_{k-4} - \dots - A_k V_0), \qquad (44)$$

where k denotes even numbers. Thus (10) really does not arise in a pure form in this expansion. Notice that, besides L, powered terms like C^3 and CH are also present in these local heat invariants. Thus the proposed rudimentary elimination contributes new proper powered terms to the ones already present there. Since the degrees defined by the considered two functions are different, these complications create new problems for the proposed rudimentary elimination. This phenomenon more emphatically underlines the inappropriateness of the proposed rudimentary elimination.

Also note that the expansion by normalized density represents these volumes $Vol_k(B_r(p))$, where $k \geq 0$, by series of the form $r(V_{k0} + V_{k2}r^2 + V_{k4}r^4 + V_{k6}r^6 + \dots)$ and such $O(r)$-series appears just regarding the first function $\int_B P_1^D(C^2, H)dvol_B$ of $a_2(p, r)$. The other two terms define $O(\frac{1}{r})$- and $O(\frac{1}{r^3})$-series. The r^3-component regarding the first function with respect to normalized-density expansion is a constant time of $CP_1(C^2, H)$, while this function is $LP_1(C^2, H)$ regarding the r^7-component.

The above arguments can easily be established also for natural expansions. Then function $vol(B_p(r))$ appears in the form described in (43). That is, no proper powered terms appear in this case which makes these computations more transparent than they were in case of normalized density expansions. The natural expansion of the functions defined on the boundary ∂B can be established by expanding also θ and multiplying this series with the AM&S-series. Since functions P_2^D and P_3^D have singularities of order $O(\frac{1}{r})$ and $O(\frac{1}{r^2})$ respectively, furthermore, $\int_B P_2^D dvol_B$ involves integration regarding dr, this Taylor series can be written in the form $\sum_k (Q^D_{1(k-4)} + Q^D_{2(k-2)} + Q^D_{3k})r^{n-4+k}$, where Q^D_{ik} is associated with the product of P_i^D and θ. That is: $Q_1^D = r^n(Q_{10}^D + Q_{12}^D r^2 + .., Q_2^D = r^{n-2}(Q_{20}^D + Q_{22}^D r^2 + .., \text{ and } Q_3^D = r^{n-4}(Q_{30}^D + Q_{32}^D r^2 + ...$ The above described r^3-coefficients appear then in terms of $Q_{12}^D(C^3, CH)$, $Q_{24}^D(CH, L, Tr(R'_u R'_u))$, and $Q_{36}^D(C^3, CH, L, Tr(R'_u R'_u))$ where, comparing with those in $[P_i^D]_3$, the arguments show up in completely new linear combinations. Indeed, the product of functions P_i^D with Θ provides different contributions to the arguments of the individual terms labeled by $i = 1, 2, 3$. Unlike in case of normalized density expansions, no powered terms show up in the natural expansion formulas. But, because of the different degrees, the proposed rudimentary elimination still remains a completely inappropriate operation also regarding this expansion.

Other explanations for the spectral indeterminacy of $||\nabla R||^2$. In this section we describe two new interrelated constructions which also prove the investigated spectral indeterminacy.

(1) In the previous constructions the required elimination of L and the other higher order density terms can properly be performed just by adjusted volume functions. More precisely, this operation can be completed by the adjusted standard volume, $Vol_0^{(adj,\rho^{-4})}$ and two higher order k-volumes, $Vol_1^{(adj,\rho^{-4})}$ and $Vol_2^{(adj,\rho^{-4})}$. One can go in the other way around, by adjusting $a_2(p, r)$ to the volume functions Vol_0, Vol_1, and Vol_2. This adjusted

function is then

$$a_2^{(adj,1,r^2,r^4)}(B) = \frac{1}{360}\Big(\int_B P_1^D(C^2,H)dvol_B + \tag{45}$$

$$+ \int_{\partial B} (r^2 P_2^D(..) + r^4 P_3(..))dvol_{\partial B}\Big), \tag{46}$$

where functions P_1, P_2, and P_3 are multiplied with 1, r^2, and r^4 respectively. This adjustment transforms the local heat invariant function to a spectrally undetermined function, which, like the volume function $vol(B(r))$, is of order $O(r)$ in case of normalized density expansions and of order $O(r^n)$ in case of natural expansions. In each cases the L and the third order power terms defined on the boundary ∂B by the last two integrals involving P_2 and P_3 appear with the same degree as L does regarding $vol(B(r))$. Also notice that the first integral $\int_B P_1^D(C^2,H)dvol_B$ defines term $LP_1(C^2,H)$ regarding this degree. Term $Tr(\nabla_u R_u \nabla_u R_u)$ present in this coefficient of $a_2^{(adj,1,r^2,r^4)}(B)$ can be obtained then by eliminating all these density terms by means of k-volume functions $Vol_0, Vol_1, Vol_2, Vol_3, \ldots$. Because of term $LP_1(C^2,H)$, this process can be completed by using more higher order volume functions than those used in the previous cases. Among these functions, which set includes also $a_2^{(adj,1,r^2,r^4)}(B)$, only Vol_0 is spectrally determined. Thus also this representation defines $Tr(\nabla_u R_u \nabla_u R_u)$ as a spectrally undetermined function. The same statement can be established by the adjusted functions $a_2^{(adj,q_1,q_2r^2,q_3r^4)}(B)$, where q_1, q_2, q_3 are arbitrary constants.

(2) The above construction strongly suggests that the spectral determinacy of $||\nabla R||^2$ can be established by splitting $a_2(B_p(r))$ into 3 different functions. The decomposition of $a_2(p,r)$ into three different parts is one of the most important characteristic features of these functions. The components in the coefficient $(Q_{12}+Q_{24}+Q_{36})$ form a linearly independent system of linear functions depending on the variables C^3, CH, L, and $Tr(R'_u R'_u)$. System $\{L, Q_{12}, Q_{24}, Q_{36}\}$ consists of for independent linear functions of these four variables, which appear as the corresponding Taylor coefficients of functions $\{a_0(p,r), Q_1(p,r), Q_2(p,r), Q_3(p,r)\}$, respectively. By the given spectral data only 2 functions:

$$\{a_0(p,r), \sum_k (Q_{1(k-4)}(p) + Q_{2(k-2)}(p) + Q_{3k}(p))r^{n-4+k}\} \tag{47}$$

are determined by which no splitting of the second function can be established. It is the minimal data which provides the desired spectral determinacy of $||\nabla R||^2$.

The same statements are true also for the spectral identities obtained by evaluating the integrals. That is, identity

$$\int (Q_{12}+Q_{24}+Q_{36})du = (\tilde{Q}_{12}(Scal^3, Scal||R||^2) + \tag{48}$$
$$+\tilde{Q}_{24}(Scal^3, Scal||R||^2, \frac{7}{2}\hat{R}-\mathring{R}, ||\nabla R||^2) +$$
$$+\tilde{Q}_{36}(Scal^3, Scal||R||^2, \frac{7}{2}\hat{R}-\mathring{R}, ||\nabla R||^2))$$

is the sum of 3 linearly independent identities which define a spectrally determined identity just in this linear combination. These component identities together with spectral identity $\tilde{L}(Scal^3, Scal||R||^2, \frac{7}{2}\hat{R}-\mathring{R}, ||\nabla R||^2)$ also determine a linearly independent system. The latter one is available now by the expansion of a_0^D. The desired spectral determinacy of can be pointed out just by a drastically changed $SIS(p)$ which, along with $\tilde{L}(Scal^3, Scal||R||^2, \frac{7}{2}\hat{R}-\mathring{R}, ||\nabla R||^2)$, separately contains all above 3 spectral identities. That is, in case of geodesic balls, the split of the heat invariants into three parts provides a new spectral data by which the desired spectral determinacy can be established. Note that this is the minimal data by which the investigated problem can be positively answered. All attempts must fail unless they are able to establish this division of the heat invariant into 3 separate invariants. But this split is certainly not encoded into the spectra of local geodesic balls, thus, all these attempts must fail altogether.

Unified spectral data. Note that, regarding the two boundary conditions, the first two components in the above decompositions are the same, thus the difference eliminates both ones, leaving behind just $P_3^D - P_3^N$. Therefore, it seems to be possible to squeeze out some kind of positive answer regarding the spectral determinacy of $||\nabla R||^2$ by eliminating $\tilde{L}$, $Scal^3$, and $Scal||R||^2$ by using the two conditions together such that the unified data $\{a_k^D(p,r), a_k^N(p,r)\}$ is extended by functions $Vol_k(p,r)$. But also this elimination is just rudimentary which does not work out by the very same reason explained earlier. In all of these cases proper elimination can be defined just by the refined spectral data $FineSIS_B(p) = \oplus_k SIS_{B,k}(p)$ which is the direct sum of $SIS_{B,k}(p)$'s defined for radial degrees $k = 0, 1, 2, 3, \ldots$. This definition excludes eliminations of terms with other ones which have different radial degrees. The positive conclusion can not be squeezed out even if the data is enlarged by the spectra of local geodesic spheres. In this case functions $\int \theta^k(p,r)du_p$ exhibit contradictory properties regarding these two different type of spectra. Actually they are incompatible which is indicated by the distinct factors $(4\pi t)^{-n/2}$ and $(4\pi t)^{-dim(S)/2}$ appearing in the

asymptotic expansions of the partition functions, furthermore, the radial degrees associated with the curvature terms are always even in case of geodesic spheres and odd in case of geodesic balls. Because of these differences, no proper eliminations of terms appearing in $FineSIS_B(p)$ can be defined by the terms appearing in $FineSIS_S^{Ex}(p)$. Thus the proper spectral data considered for the unification of the ball-spectra with the sphere-spectra must be defined by the direct sum $FineSIS_S(p) \oplus FineSIS_B(p)$.

Summary 4.1. (A) Case of spheres. *On harmonic manifolds the Lichnerowicz identity $\mathcal{L}i(p)$ is not encoded into the space, $SIS^{ex}(p)$, of spectrally determined identities defined for the spectra of local geodesic spheres. Since this space, defined by the radial expansion coefficients of functions $a_k(p,r)$ and $\int \theta^k(u_p,r)du_p$, is identical to the complete space of spectrally determined identities, the Lichnerowicz identity is not determined by the given spectral data, that is, it can not be computed by the spectra of local geodesic spheres. So is any identity $\mathcal{F} = \mathcal{S} + Q\mathcal{L}i$, where $\mathcal{S} \in SIS^{ex}(p)$ and $Q \neq 0$. In [AM-S], such spectrally undetermined identity is used to compute $||\nabla R||^2(p)||^2$.*

The greatest imperfection of $SIS(p)$ is that the radial expansion coefficients $\int F_c(u)du_n$, whose integral-evaluation provides the curvature expressions, are considered without any reference to the radial degree c, thus this data can not be used for constructing radial functions. (Such radial constructions can be implemented just by functions $\int F_c(u)du_n r^c$. Also note that, in case of geodesic spheres, this degree is even.) As it is pointed out above, eliminations using coefficients having different radial degrees result, on one hand radial functions from which the considered term is not eliminated, and, on the other hand, the curvature expression resulting by the formal elimination does not appear as an expansion coefficient of a function being in $Span\{a_k(p,r)\}$ or $Span\{a_k(p,r), \int \theta^k(u_p,r)du_{np}\}$. In order to control this contradictory situation, refined spectral data $FineSIS(p) = \oplus_c SIS_c(p)$ resp. $FineSIS^{ex}(p) = \oplus_c SIS_c^{ex}(p)$, where $SIS_c(p)$ resp. $SIS_c^{ex}(p)$ consists of coefficients having the same radial degree c should be introduced to the computations. This data is defined after choosing the same type of expansion for each function from the set $\{a_k(p,r)\}$ resp. $\{\int \theta^k(u_p,r)du_{np}\}$. The direct sum applied in these definitions excludes the usage of coefficients of different radial degrees during eliminations. Due to the identity $\int r^{2d}\theta^k(u_p,r)du_p = r^{2d}(\int \theta(u_p,r)du_{np})^k$ valid on harmonic manifolds, a larger, spectrally determined data $FineSIS^{ex}(p) \subset FineSIS^{Ex}(p)$ can be defined by the elements of the larger function space

$$Span\{a_k(p,r), \int \theta^k(u_p,r)du_{np}\} \subset Span\{a_k(p,r), \int r^{2d}\theta^k(u_p,r)du_{np}\}.$$

Although (10) and the investigated r^2-coefficients of $a_2(p,r)$ appear with different degrees, they can be brought together to have the same radial degree by the above degree-adjusting operations. Also note that the Lichnerowicz identity (which is not derived from a radial function $l(p,r)$, thus no radial degree can be defined for it) can not be built into the space $FineSIS(p)$. By the above arguments, it does not show up even in the rudimentary $SIS(p)$. Therefore, the ultimate reason for $||\nabla R||^2(p)||^2$ is not determined by the spectra of local geodesic spheres is that it can be computed just by the Lichnerowicz identity.

(B) Case of balls. *In the ball case, the AM&S-paper proposes elimination of L occurring in the r^3-coefficient of $a_2(B_p(r))$ by the L occurring in the r^7-coefficient of the spectrally determined function $vol(B_p(r))$. Since these terms are defined for different radial degrees, by the above explanation, this rudimentary elimination can not be applied for these computations. An other problem is that the considered r^3-coefficient contains higher order powered terms which also must be eliminated. For this operation one must use also the proper k-volumes, $Vol_k(p,r) = \int_{B_p(r)} \Theta_p^k dB_p(r)$, defined for $k \geq 1$. Since these volumes are not determined by the standard volume $vol(B_p(r)) = Vol_0(p,r)$, they are not determined by the Dirichlet or Neumann spectra of balls either. Due to these facts, powered identities associated with $\int C^3(u)du_n$ and $\int C(u)H(u)du_n$ become spectrally undetermined. As a result, also $||\nabla R||^2$ becomes spectrally undetermined.*

Proper elimination of these terms can be implemented just by using the adjusted volumes $Vol_k^{adj,\rho^{-4}}(p,r) = \int_{B_p(r)} \rho^{-4}\Theta_p^k dB_p(r)$, where $k \geq 0$. But these adjustments further worsen the spectral indeterminacy of the investigated terms. Even $Vol_0^{adj,\rho^{-4}}(p,r)$ becomes a spectrally undetermined quantity, meaning, that the spectral determinacy of L (regarding the spectra of local geodesic balls) is a relative concept. It depends on the degree of the radial expansion coefficient in which it shows up. In the considered r^3-coefficient it is a spectrally undetermined term and so are the other powered density-terms occurring there. This is an other proof of the spectral indeterminacy of $||\nabla R||^2$. These arguments also show that, in case of geodesic balls, extensions $FineSIS^{ex}(p)$ or $FineSIS^{Ex}(p)$ provide spectrally undetermined quantities to the elements of $FineSIS(p)$. This is the most drastic difference between the sphere- and ball-cases.

Quantity $||\nabla R||^2$ can be determined just by appropriate extended data. Several of them are constructed above. All of them contain spectrally undetermined identities which play the very same role than what the Lichnerowicz identity does when space $SIS^{ex}(p)$ of spectrally determined identities is ex-

tend by the Lichnerowicz identity in order to determine $||\nabla R||^2$*. It is also pointed out above that this quantity is not determined even by any of the unified spectral data.*

5 Rudimentary uncertainty on Riemann manifolds.

The above considerations exhibit strong physical contents. Namely, one can explain by the given spectral data that how does the uncertainty principle manifest itself on Riemannian manifolds. Here we consider only the rudimentary case. This idea works out on general Riemannian manifolds, however, by using the available formulas, we start to explain it on harmonic manifolds. Despite this analogy, it should be pointed out that this theory is pure geometric which can not be called physical, by any means. Although the constants and and relations associated below with spectrally undetermined identities (such as the Lichnerowicz identity) strongly remind the Planck constant and the corresponding Heisenberg relation, no physical interpretation for these identities exist at this point. This is why this theory is not associated with any kind of physics at this point. It has no apparent relation even to the exact physical models described in the last section.

The constants and relations mentioned above appear on the scene when the distance of identities (such as the Lichnerowicz identity) from $SIS(p)$ is computed. These computations are carried out in a Hilbert space in the following way. Consider all pre-functions $F(u)$ defined in terms of arbitrary linear combinations of radial curvature terms appearing in the AM&S-expansions. In the expansions these terms appear just in particular combinations but in generalized situations all linear combinations are considered. The integrals of functions $F(u)$ define a linear space of equations endowed with the pre-Hilbert inner product defined by integrating products $F_1(u)F_2(u)$ by du. By topological closure, one defines an extended identity Hilbert-space $EIS(p)$. Now let $P(\mathcal{L})$ be the spectrally determined identity obtained by projecting the Lichnerowicz identity onto $SIS(p)$. It can be obtained by integrating the projected function $P(L(u))$ by du. The identities in $SIS(p)$ are interpreted such that they can be clearly heard by the given spectral data while the other identities in $EIS(p)$ which are in the complement of $SIS(p)$ can be heard just together with noise. Such noisy identity is the Lichnerowicz identity, for which the noise wave is defined by $NW(L(u)) = L(u) - P(L(u))$. The magnitude of noise is defined by the L^2-norm $||NW(L(u))||^2 = LH(||\nabla R||^4, ..., \hat{R}\mathring{R}, ...)$ which is equal to an expression written in terms of the curvature terms appearing in the equations. Like

in the physical Heisenberg relations, these terms are typically products of those appearing in the primary equation $\mathcal{NW}(L)$. Note that these quantities are defined by the difference $L(u) - P(L(u))$, which makes the reminiscence with the Heisenberg relations even stronger. The Lichnerowicz-Planck constant, $h_{\mathcal{L}}$, is defined by the minimum of values $||NW(L(u))||^2$ considered for all $L(u)$ whose integration provides $\mathcal{L}$. Then the Lichnerowicz-Heisenberg relation is

$$||NW(L(u))||^2 = LH(||\nabla R||^4, ||\nabla R||^2 \hat{R}, ||\nabla R||^2 \hat{R}, \hat{R}\mathring{R}, ...) \geq h_{\mathcal{L}}, \qquad (49)$$

which clearly describes the positive distance of $\mathcal{L}$ from the spectrally determined identities.

A more general version of these relations can be defined such that one replaces $P(L(u))$ by an arbitrary other function $F(u) \in Pre(SIS(p))$, by which the noise wave, $NW_F(L(u))$, is defined in the same way as by means of $P(L(u))$. Quantity $||NW_F(L(u))||^2$ measures the magnitude of noise when the Lichnerowicz identity is listened by $F(u)$. Formally this function looks the same as the previous one, but the noise is going to be bigger. That is, the left side of the general HL-relation $||NW_F(L(u))||^2 \geq h_{\mathcal{L}}$ usually gets bigger in this situation.

These objects can be defined on arbitrary Riemann manifolds for any identity. The spectral data can be defined by local geodesic spheres as well as balls. In order to avoid the above discussed problems, the AM&S-expansions must be established by the non-normalized density. Hilbert spaces $SIS(p)$ and $EIS(p)$ can be defined in the same way as they were on harmonic manifolds. It is also well known that Lichnerowicz established his identity for general Riemannian manifolds, where it appears in a much more complicated form, containing additional terms beyond those appearing on Einstein manifolds. Anyhow, one can prove also in these most general cases that this general identity is not encoded into the the spectra of local geodesic spheres or balls. More precisely we have:

Summary 5.1. *On a general Riemann manifold the Lichnerowicz identity, considered at a point p, is not determined by the spectra of local geodesic spheres or balls concentrated at p. That is, this identity is not encoded into the spectral identity space $SIS(p)$ but it rather shows up as a spectrally undetermined equation in $EIS(p)$. This fact can be explained also such that prefunctions defining the Lichnerowicz identity do not show up in $Pre(SIS(p))$ but in the larger pre-space $ExPre(SIS(p)) = Pre(EIS(p))$ where they have strictly positive distance from $Pre(SIS(p))$. The corresponding noise waves,*

Plank constants, and Heisenberg relations can explicitly be computed also in these most general situations.

This mathematical uncertainty theory matches the physical one, by all means. A curvature identity $\mathcal{F}$ defined by the integral $\int F(u)du_n$ of a pre-function is always a true identity. This logical value is independent from its relation to the spectral identity space $SIS(p)$. The above quantities measure that in what extend can this well determined identity be measured (or, recovered) by the given spectral data. It is also clear that spectrally undetermined identities come from two sources. Either by powering spectrally determined identities, or by considering such identities like Lichnerowicz's which are nothing to do with pre-functions $F(u) \in Pre(EIS(p))$.

Several alternative versions of the above concepts can be introduced as follows. For establishing the first one, consider a geodesic sphere or ball of radius $R(p)$ about the center p. In the following definition the data can either be the spectra of local geodesic spheres or balls concentrated at p. If the Taylor expansion $a_k(p,r) = \sum_{s=0}^{\infty} \frac{1}{s!} A_{ks}(p) r^s$ of the local heat invariants is convergent for $r = R(p)$, then, for any fixed k, heat invariant $a_k(p,R) = \sum_{s=0}^{\infty} \frac{1}{s!} A_{ks}(p) R^s$ (of the sphere or ball of radius $R(p)$) can be considered as a curvature identity, $\mathcal{A}_k(R(p))$, defined as an infinite sum (series) of identities $\frac{R^s}{s!} A_{ks} \in SIS(p)$ determined for the coefficients of this expansion. By this construction, one can associate the identity space $SIS(R(p)) = \sum_{k=0}^{\infty} \mathcal{A}_k(R(p))$ to the spectrum of any fixed geodesic sphere or ball, which is just a very thin subspace of the total space $SIS(p)$. For a field of geodesic spheres or balls, having radius $R(p)$ at a point p, the $SIS(R(p))$ defines a Hilbert space bundle over the manifold, for which the Planck constants and Heisenberg relations can be defined in the same way as for $SIS(p)$. If an identity is not determined by $SIS(p)$, it is even so regarding $SIS(R(p))$. This construction shows that the information encoded into the spectral data is considerably weakened if, instead of all local geodesic spheres or balls concentrated at p, it is provided only by a single sphere or ball having center p.

Other concepts analogous to the AM&S local heat invariants are the so called *remote local heat invariants*, $\rho_k(p,\tau)$, defined at a point p of a compact Riemann manifold by means of the cut locus $Cut(p)$. For a unit vector $u_p \in T_p(M)$, let $q(u_p) \in Cut(p)$ be the point such that the geodesic starting out from p into the direction of u_p intersects $Cut(p)$ at $q(u_p)$. Parameterization, τ, on this geodesic, $c_{pq}(\tau)$, is chosen such that $c_{pq}(0) = p, c_{pq}(1) = q(u)$, furthermore, the speed vector $\dot{c}_{pq}(\tau)$ has constant length equal to the arc-length of c_{pq}. This parameter is called cut-locus-radius function. For a fixed

$0 \leq \tau \leq 1$, cut-locus-sphere $CuS_p(\tau)$ of center p and radius τ is defined by consisting of points which have parameter τ on each of the above geodesics. Cut-locus-balls, $CuB_p(\tau)$ are defined by points having parameters $\leq \tau$. The remote local heat invariants are defined by the cut-locus-radial expansion (i. e., τ-expansion) of the heat invariants $a_k(CuS_p(\tau))$ resp. $a_k(CuB_p(\tau))$ defined on cut-locus-spheres resp. cut-locus-balls. Spectral identity spaces $SIS(Cut(p))$ and $SIS(Cut(\tau(p)))$, where τ denotes fixed cut-locus-radius, regarding cut-locus-spheres resp. balls can be introduced in the same way as for geodesic spheres resp. balls.

The above constructions allow to introduce spectral identity spaces also for compact sub-domains, D, with boundaries, ∂D, of Riemannian manifolds M^n. For the sake of simplicity suppose that ∂D is diffeomorphic to the Euclidean unit sphere S^{n-1}, furthermore, let $p \in D$ be a point such that for any $q \in \partial D$ there exists a unique geodesic $c_{pq} \subset D$ joining the two points. On such a geodesic, one can define the same parameterization, τ_p, satisfying the very same properties described above. By replacing cut locus $Cut(p)$ with boundary ∂D in the above constructions, one can define spectral identity space $SIS(p, \partial D)$ (where ∂D appears as level set $\tau_p = 1$) regarding the spectrum of Laplacian defined on ∂D or any of the spectra defined by particular boundary conditions on D. By continuous movement of p, one can define this identity space together with the associated Planck constants and Heisenberg relations at any point lying in the interior of D. Both the spectrally determined and undetermined identities can precisely be described in terms of the Hilbert space bundles established in this construction.

6 Global vs. local spectral investigations.

On compact Riemann manifolds, M, the integrals of the infinitesimal heat invariants define the so called averaged infinitesimal heat invariants. The exploration of relations between the global spectra and the averaged infinitesimal heat invariants seems to be an interesting question. In this paper we establish just a single theorem concerning the relation between the global spectra and the volume of small geodesic balls and spheres. Next we prove that, contrary to the volume of the whole compact manifolds, these local averaged volumes are not determined by the spectrum of M.

By (2), the fourth coefficient appearing in the radial expansion of $\mathbf{A}^k_{c(r)}$ is a linear combination of terms $TrR^2_{u_p}$ and $TrR''_{u_p} = Ricc(u_p, u_p)|u_p|u_p$, where $|u_p|u_p$ denotes two covariant differentiations regarding u_p. Integration of TrR''_{u_p} regarding du_p at a fixed points provides linear combinations of terms

like $\Delta(Scal)(p)$ and $(Ricc^{ab}_{|a})_{|b}$, whose integrals vanish on the whole compact manifold, by the Stokes theorem. The same integral applied to the first term provides a constant time of $||Ricc||^2 + (3/2)||R||^2$.

To establish the statement we consider also the one parametric families $(G, g_{\lambda(t)})$ of 0-isospectral manifolds constructed by Schüth [S] on compact Lie groups G such as:

$$SO(m) \times T^2 \ (m \geq 5),\ Spin(m) \times T^2 \ (m \geq 5),\ SU(m) \times T^2 \, (m \geq 3), \quad (50)$$

$$SO(m) \, (m \geq 9),\ Spin(m) \, (m \geq 9),\ SU(m) \, (m \geq 6),\ SO(8),\ Spin(8). \quad (51)$$

She proved that distinct members in a one-parametric family have distinct spectra on 1-forms by showing that they have Ricci tensors of different norm. We use the very same computational technique (she applied to prove this latter statement) for establishing our statement.

Consider $a_2(G, g_{\lambda(t)})$ for a family, which, due to the isospectrality, is constant regarding t. Since $Scal$ is constant on G, furthermore, $vol(G)$ and $\int_G Scal$ are spectrally determined, also the first term, $\int_G Scal^2$, in $a_2(G, g_{\lambda(t)})$ is constant regarding t. It follows then that the averaged volumes of small geodesic balls and spheres can not be constant, otherwise the above fourth coefficient in the expansion of the density together with $a_2(G, g_{\lambda(t)})$ define a non-degenerated system of linear equations which would determine constant values both for $||Ricc||^2(t)$ and $||R||^2(t)$. This idea works out for all invariants $Tr\mathbf{A}^k_{c(r)}$. By summing up we have

Theorem 6.1. *The averaged volumes of small geodesic balls and spheres are generically not determined by the spectra of compact Riemann manifolds. This spectral indeterminacy is exhibited by the one parametric families $(G, g_{\lambda(t)})$ constructed by Schüth [S] on the compact Lie groups G listed in (50) and (51). The fourth coefficient in the power series expansion of the volumes of small geodesic balls resp. spheres depends on parameter t. This coefficient is expressed in terms of $||Ricc||^2$ and other lower order curvature invariants. Schüth originally proved that quantity $||Ricc||^2$ is different on the distinct members of a family, therefore, she concluded, they are not isospectral on 1-forms.*

This statement can be established, by the same proof, for $Tr\mathbf{A}^k$-volumes of geodesic balls and spheres where the integration can be defined either with θdu or just with du. This statement is highly expectable also on the Gordon-Wilson examples [GW] of isospectral manifolds having different local geometries as well as on those derived from these examples. Since these manifolds have non-constant scalar curvature in general, this version of the

theorem can not be established by using only the fourth coefficient of the above power series expansion.

7 Isospectralities and physical symmetries.

Both my isospectrality examples and the Gordon-Wilson examples [GW] (together with those derived from the GW-constructions, see a survey on them in [S]) arise from 2-step nilpotent Lie groups. Yet, in terms of spectral stability of small geodesic balls and spheres these two types of examples are the polar opposites of each other. One of the idiosyncrasies of the GW-examples and their relatives is the wide range of changing invariants during the continuous isospectral deformations introduced in their constructions. Whereas, these quantities are not changing during the discrete isospectral deformations introduced in my constructions. In my isospectrality families the members share even the same volume function $\theta(r)$, therefore, also the volumes of geodesics spheres resp. balls having the same radius must be identical. Thus the above theorem is a clear demonstration of these arguments. Later, after introducing the technical definitions, it is more clearly pointed out that what causes these significant differences between these two different type of constructions.

The isospectrality exhibited by my examples has a deep physical meaning. Namely, it is equivalent to the C-symmetry known in quantum theory. This physical connection is completely established in [Sz8]. Actually, the bulk of this whole section is going to be an outline of some of the results developed in [Sz8]. Readers interested in details may consult with this paper, however, the following review tends to be self-contained as much as it is possible.

Before going into technical details, we briefly describe this physical connection by non-technical words. As it is pointed out below, on the Riemann manifolds used for these constructions the Laplacian is nothing but the quantum Hamilton operator of particle systems described in elementary particle physics. The action of the intertwining operator constructed for establishing the isospectrality can be interpreted such that it exchanges some of the particles for their anti particles. Thus the mathematically established isospectrality can be paraphrased such that the spectrum of an elementary particle system, that is, the possible energy levels on which the system can exist, is not changing if some of the particles are exchanged for their anti particles.

This paraphrase is a clear manifestation of the C-symmetry principle

introduced in physics in the following general form: "The laws are the same for particles and antiparticles." It should be emphasized, however, that this general principle will be pointed out only regarding the spectrum of the Hamilton operator. In other words, the isospectralities exhibited by my examples are clear manifestation of the *spectral C-symmetry principle.*

A preliminary description of the endomorphisms exchanging the particles for their antiparticles is as follows. The two-step nilpotent Lie groups used for these constructions are defined on the (X,Z)-space, $\mathbb{R}^k \times \mathbb{R}^k = \mathcal{X} \times \mathcal{Z}$ by a linear space, $E_{skew}(\mathcal{X})$, of skew endomorphisms acting on the X-space. The members of an isospectrality family are defined on the same (X,Z)-space by a corresponding family of endomorphism spaces where for any two members, $E_{skew}(\mathcal{X})$ and $E'_{skew}(\mathcal{X})$, there exists an involutive orthogonal transformation $\sigma : \mathcal{X} \to \mathcal{X}$ commuting with all these endomorphisms and the endomorphisms belonging to $E'_{skew}(\mathcal{X})$ can be obtained by composing the elements of $E_{skew}(\mathcal{X})$ by σ, that is, $E'_{skew}(\mathcal{X}) = \sigma \circ E_{skew}(\mathcal{X})$. These conditions imply the existence of an orthogonal direct sum $\mathcal{X} = \mathcal{X}^{(a)} \oplus \mathcal{X}^{(b)}$ where both subspaces are invariant under the actions of all endomorphisms and the first component is fixed under the action of σ, while it is $-id_{\mathcal{X}^{(b)}}$ on the orthogonal complement $\mathcal{X}^{(b)}$. It is explained later that these endomorphisms serve as angular momenta for particles orbiting in $\mathcal{X}^{(a)}$ resp. $\mathcal{X}^{(b)}$. These endomorphisms reverse the angular momentum for particles living on $\mathcal{X}^{(b)}$ and preserve it for those orbiting in $\mathcal{X}^{(a)}$. The operators intertwining the Laplacians are defined by means of these exchange-endomorphisms. Let it be mentioned yet that these constructions allow just discrete isospectral deformations.

By using the same two-step nilpotent Lie group in two different ways, these examples are constructed on two different type of manifolds. The first ones are torus bundles defined over the X-space by factoring the center, $\mathcal{Z}$, by a Z-lattice Γ_Z, and the others are ball resp. sphere bundles over the X-space defined by considering appropriate Z-balls resp. Z-spheres in $\mathcal{Z}$. For the sake of simplicity, next we describe only the ball-bundle cases. The spectral investigations must be established on the function spaces defined on these manifolds. Actually both function spaces can be considered on the same non-factorized manifold. Namely, in the first case, it is the space of Γ_Z-periodic functions, while in the second case it is the function space satisfying an arbitrarily prescribed boundary condition on the boundary manifold which is a Z-sphere bundle. Both function space can be written up by appropriate Z-Fourier transforms. In the first case it is the discrete Z-Fourier transform defined by the Z-lattice. The big advantage of this transform is that it separates the X- and Z-variables from each other. The

other Z-Fourier transform introduced for investigating Z-ball bundles is the so called twisted Z-Fourier transform. This name indicates that this is a much more complicated version of the Z-Fourier transforms where the X- and Z-variables can not be separated from each other. This transform involves mixed, so called twisting functions which establish the fundamental bonds between the X- and Z-spaces.

One of the most important observation regarding these two exact mathematical models is that with the help of them the electromagnetic, the weak-, and the strong-nuclear forces can be established within a unified framework. The main unifying principle is that these forces can be described by the eigenfunctions of the very same Laplacian such that the distinct forces emerge on distinct invariant subspaces of this common quantum operator. This very same Laplacian is the Laplacian acting on the Lie group and the two distinct function spaces are the Γ_Z-periodic function space resp. those defined for Z-ball bundles. After its action on the Fourier integral formula, the very same Laplacian appears in drastically different ways behind the integral sign. In the first case, it turns into a Ginsburg-Landau-Zeeman operator of charged particles orbiting in complex planes in magnetic fields perpendicular to the planes, while, in the second case, it appears as a sum of a scalar operator which can be identified as the quantum Hamilton operator of electroweak interactions and a new type of spin operator which can be identified as the quantum Hamilton operator of strong force interaction keeping the nucleus together. The particles attached to the two function spaces are distinguished by calling them particles having no interior resp. those having interior.

The Hamilton operators are defined in this theory on the nilpotent groups. The corresponding wave and Schrödinger operators emerge in the Laplacians of the static resp. solvable extensions of these nilpotent groups. The latter manifolds are endowed with a natural invariant indefinite metric of Lorentz signature. Thus, these new exact mathematical models provide a relativistic theory for elementary particles. The above discussed functions defined by Z-Fourier transforms appear in the explicit solutions of these wave operators. These wave functions strongly remind those introduced by de Broglie in classical wave mechanics (cf. [P], volume 5). Actually, they are the only appropriate adaptations of the original de Broglie wave functions to the new mathematical models. They carry over also the original de Broglie theory to understand the much more complicated physical situation inherent in the new models. This theory establishes infinitely many non-equivalent models for which even classification is possible. The particle systems attached to them behave exactly like those introduced by the

familiar standard model of elementary particle physics.

7.1 Technicalities on 2-step nilpotent Lie groups.

A 2-step nilpotent metric Lie algebra, $\{\mathcal{N}, \langle , \rangle\}$, is defined on a real vector space endowed with a positive definite inner product. The name indicates that the center, $\mathcal{Z}$, can be reached by a single application of the Lie bracket, thus its second application always results zero. The orthogonal complement of the center is denoted by $\mathcal{X}$. The Lie bracket operates among these subspaces according to the following formulas:

$$[\mathcal{N}, \mathcal{N}] = \mathcal{Z} \quad , \quad [\mathcal{N}, \mathcal{Z}] = 0 \quad , \quad \mathcal{N} = \mathcal{X} \oplus \mathcal{Z} = \mathbb{R}^k \times \mathbb{R}^l. \tag{52}$$

Spaces $\mathcal{Z}$ and $\mathcal{X}$ are called Z- and X-space, respectively.

Up to isomorphisms, such a Lie algebra is uniquely determined by the linear space, $J_{\mathcal{Z}}$, of skew endomorphisms $J_Z : \mathcal{X} \to \mathcal{X}$ defined for Z-vectors $Z \in \mathcal{Z}$ by the formula

$$\langle [X, Y], Z \rangle = \langle J_Z(X), Y \rangle, \forall Z \in \mathcal{Z}. \tag{53}$$

This statement means that for an orthogonal direct sum, $\mathcal{N} = \mathcal{X} \oplus \mathcal{Z} = \mathbb{R}^k \times \mathbb{R}^l$, a non-degenerated linear map, $\mathbb{J} : \mathcal{Z} \to E_{skew}(\mathcal{X})$, $Z \to J_Z$, from the Z-space into the space of skew endomorphisms acting on the X-space, defines a 2-step nilpotent metric Lie algebra on $\mathcal{N}$ by (53). Furthermore, an other non-degenerated linear map $\tilde{\mathbb{J}}$ having the same range $\tilde{J}_{\mathcal{Z}} = J_{\mathcal{Z}}$ as $\mathbb{J}$ defines a Lie algebra which is isomorphic to the previous one. If isometric isomorphism is required, then $\tilde{\mathbb{J}}^{-1} \circ \mathbb{J}$ must be an orthogonal transformation on the Z-space.

With the help of these technical definitions one can more clearly explain why do the GW-type constructions produce completely different examples from those I constructed by the above exchange-endomorphism σ. In the GW-case the isospectral deformation is defined such that, for any fixed Z, the spectrum of the bilinear map $\langle J_Z(X_1), J_Z(X_2) \rangle$ (which defines a unique linear map on the Euclidean X-space) is not changing, but, the spectrum of $\langle J_{Z_1}(X), J_{Z_2}(X) \rangle$ (defined, now, on the Euclidean Z-space for an arbitrary fixed X) is changing. Other characteristic features are that these maps have different eigenvalues, furthermore, the Z-space has dimension 2. The changing spectrum of $\langle J_{Z_1}(X), J_{Z_2}(X) \rangle$ gives rise to the wide range of changing invariants (including also spectral invariants) defined for Z-balls, Z-spheres, geodesic balls, and geodesic spheres. This is the ultimate reason why the GW isospectrality examples are established just on Z-torus bundles but not on Z-ball and Z-sphere bundles, where they are actually non-isospectral.

Whereas, the spectra both of $\langle J_Z(X_1), J_Z(X_2)\rangle$ and $\langle J_{Z_1}(X), J_{Z_2}(X)\rangle$ are not changing during the discrete deformations I introduced for the constructions. This is why my isospectrality examples live both on Z-torus bundles and Z-ball resp. Z-sphere bundles. Due to these differences, the following formulas established in my case resist any attempt to define them on the GW-examples. In this paper we describe only such cases where the spectra of the above maps consist only the same values. Thus difficulty arises already in the first step when one tries to extend them to the GW-type constructions, where these spectra consist of different eigenvalues. But the ultimate reason for this extension breaks down is that the spectrum of $\langle J_{Z_1}(X), J_{Z_2}(X)\rangle$ is changing during the continuous deformations they introduced for their constructions. These difficulties strongly indicate that the distinguishing characteristics exhibited by these two different cases can be explored by mutually distinct methods. This field can not be explored by a single method with the help of which one would be able to control both the changing and the non-changing invariants exhibited by these two different type of constructions.

Important particular 2-step nilpotent Lie groups are the Heisenberg-type groups [K] defined by endomorphism spaces $J_{\mathcal{Z}}$ satisfying the Clifford condition $J_Z^2 = -\mathbf{z}^2 id$, where $\mathbf{z} = |Z|$ denotes the length of Z-vectors. These groups are attached to Clifford modules (representations of Clifford algebras). The well known classification of these modules provides classification also for the Heisenberg-type groups. According to this classification, the X-space of a H-type group is an $(a+b)$-times Cartesian product of a smaller space $\mathcal{Y} = \mathbb{R}^{n_l}$, which is endowed with an l-dimensional endomorphism space $j_{\mathcal{Z}}$ such that the endomorphisms acting on $\mathcal{X} = \mathcal{Y}^a + \mathcal{Y}^b$ are defined by $J_Z = j_Z \times \cdots \times j_Z \times -j_Z \times \cdots \times -j_Z$. Dimension n_l depends only on l.

By means of the exponential map, also the group can be considered to be defined on $\mathcal{N}$. That is, a point is denoted by (X, Z) also on the group. Metric tensor, g, is defined by the left invariant extension of $\langle , \rangle$ onto the group N.

Although most of the results described below extend to general 2-step metric nilpotent Lie groups, next only H-type groups will be considered. On these groups, the Laplacian appears as follows:

$$\Delta = \Delta_X + (1 + \frac{1}{4}|X|^2)\Delta_Z + \sum_{\alpha=1}^{r} \partial_\alpha D_\alpha \bullet, \tag{54}$$

where $\{e_\alpha\}$ is an orthonormal basis on the center (Z-space) and $D_\alpha \bullet$ denotes directional derivatives along the vector fields $X \to J_\alpha(X) = J_{e_\alpha}(X)$,

furthermore, $\mathbf{x} = |X|$ denotes the length of X-vectors.

In the next sections, it is described in technical terms that how this operator manifests itself as the Hamilton operator of different elementary particle systems. A non-technical preview of this physical interpretation is as follows. As it is indicated above, the main idea is that these systems are attached to various invariant function subspaces of Δ which can be divided into two major classes. The first class is defined by Z-torus bundles (alias Z-crystals), where the corresponding function space consists of functions which are periodic regarding the Z-lattice Γ_Z defining the Z-torus bundle. The particles attached to such a function spaces are considered to be point particles having no interior. They can show up at the lattice points of Γ_Z, where the angular momentum is defined for them by means of J_Z. As it will turn out, the Δ appears on this function space as the Ginsburg-Landau-Zeeman operator of charged particles. Thus the natural interpretation for these systems is that they consist of electrons, positrons, and electron-positron-neutrinos. Contrary to these cases, the other type of particles, emerging on function spaces defined on Z-ball bundles by Dirichlet or Neumann boundary conditions, do have interior which space is represented by the interior of the Z-balls. The X-space represents always the exterior word. The Laplacian appears on these Z-ball-bundles as the sum of Hamilton operators of electro-weak and strong-force interactions. A technical description of these enormous differences between Z-torus and Z-ball bundles is as follows.

7.2 Z-crystals modelling Ginsburg-Landau-Zeeman operators.

The Z-torus bundles are defined by factorizing, $\Gamma\backslash H$, a two-step nilpotent group H by a Z-lattice, $\Gamma = \{Z_\gamma\}$. The name indicates that this lattice is defined only on $\mathcal{Z}$ and not on the whole (X, Z)-space. Such a factorization defines a Z-torus bundle over the X-space. The natural Z-Fourier decomposition, $L^2_{\mathbb{C}} := \sum_\gamma W_\gamma$, of the L^2 function space is defined such that subspace W_γ is spanned by functions of the form

$$\Psi_\gamma(X, Z) = \psi(X)e^{2\pi\mathbf{i}\langle Z_\gamma, Z\rangle}. \tag{55}$$

Note that each W_γ is invariant under the action of Δ, more precisely, we have:

$$\Delta\Psi_\gamma(X, Z) = (\lhd_\gamma\psi)(X)e^{2\pi\mathbf{i}\langle Z_\gamma, Z\rangle}, \quad \text{where} \tag{56}$$

$$\lhd_\gamma = \Delta_X + 2\pi\mathbf{i}D_\gamma \bullet -4\pi^2|Z_\gamma|^2(1 + \frac{1}{4}|X|^2). \tag{57}$$

In terms of parameter $\mu = \pi|Z|_\gamma$, this operator can be written in the form $\lhd_\mu = \Delta_X + 2\mathbf{i}D_\mu \bullet -\mu^2|X|^2 - 4\mu^2$. Although it is defined in terms of the X-variable, this operator is not a sub-Laplacian which is resulted by a submersion. It should be considered as restriction of the total Laplacian onto the invariant subspace W_γ. Actually, the Z-space is represented by the constant μ and operator $D_\mu\bullet$. A characteristic feature of this restricted operator is that it involves only a single endomorphism, J_{Z_γ}.

On a Z-crystal, $B_R \times T^l$, for given boundary condition $Af'(R^2)+Bf(R^2) = 0$, the eigenfunctions of $\lhd_\mu$ can be represented in terms of n^{th}-order complex valued spherical harmonics $\mathtt{H}^{(n,m)}(X)$ in the form $f(\langle X, X\rangle)\mathtt{H}^{(n,m)}(X)$, where the radial function f is an eigenfunction of the ordinary differential operator

$$(\diamondsuit_{\mu,\tilde{t}} f)(\tilde{t}) = 4\tilde{t}f''(\tilde{t}) + (2k+4n)f'(\tilde{t}) - (2m\mu + 4\mu^2(1+\frac{1}{4}\tilde{t}))f(\tilde{t}). \quad (58)$$

Degree m is defined such that $\mathtt{H}^{(n,m)}(X)$ is simultaneously eigenfunction also of $\mathbf{i}D_\mu\bullet$ with eigenvalue $m\mu$.

Polynomials $\mathtt{H}^{(n,m)}(X)$ can be constructed as follows. Consider a complex basis $\mathbf{B} = \{Q_1, \ldots, Q_{k/2}\}$ regarding the complex structure $J_{Z_{\mu u}}$ and an n^{th}-order polynomial $\prod z_i^{p_i}(Z_{\mu u}, X)\overline{z}_i^{q_i}(Z_{\mu u}, X)$, where $z_i(Z_{\mu u}, X) = \langle Q_i + \mathbf{i}J_{Z_{\mu u}}(Q_i), X\rangle$ and $\sum(p_i + q_i) = n$. Since $z_i(Z_{\mu u}, X)$ is an eigenfunction of $\mathbf{i}D_\mu\bullet$ with eigenvalue $-\mu$, the whole polynomial is also an eigenfunction with eigenvalue $m\mu$, where $\sum(-p_i + q_i) = m$. However, this homogeneous polynomials are not harmonic regarding Δ_X and their restrictions do not define spherical harmonics on the unit X-sphere. In order to have the spherical harmonics, these spherical but non-harmonic functions should be projected into the space of n^{th}-order spherical harmonics. These projections, $\Pi_X^{(n)}$, are extensively described in my papers [Sz4]-[Sz8] (see for instance Section 6.4 in [Sz8]). They can be represented as uniquely determined polynomials of the spherical Laplacian Δ_S. Since operator $\mathbf{i}D_\mu\bullet$ commutes with this Laplacian, projection $\Pi_X^{(n)}((\prod z_i^{p_i}\overline{z}_i^{q_i}))(Z_{\mu u}, X)$ really provides a desired polynomial $\mathtt{H}^{(n,m)}(X)$. Since these projections are onto whose kernels are formed exactly by the lower order polynomials, all these polynomials can be obtained by this construction.

In the 2D-case, operator $\lhd_\mu$ can be transformed to the the well known Ginsburg-Landau-Zeeman operator [Bo, LL, P]

$$-\frac{\hbar^2}{2m}\Delta_{(x,y)} - \frac{\hbar eB}{2mc\mathbf{i}}D_z \bullet + \frac{e^2B^2}{8mc^2}(x^2+y^2) \quad (59)$$

of a charged particle orbiting in the (x, y)-plane in constant magnetic field directed toward the z-axis by choosing $\mu = eB/2\hbar c$ and multiplying the whole operator with $-\hbar^2/2m$. For k-dimensional X-spaces, number $\kappa = k/2$ means the *number of particles*, and, endomorphisms j_Z and $-j_Z$ in the above formulas are attached to systems electrons resp. positrons. More precisely, by the classification of H-type groups, these endomorphisms are acting on the irreducible subspaces $\mathbb{R}^{n_l}$ and the system is interpreted such that there are $n_l/2$ particles of the same charge orbiting on complex planes determined by the complex structures j_{Z_u} resp. $-j_{Z_u}$ in constant magnetic fields.

Note that this operator contains also an extra constant term $4\mu^2$, which, after establishing the corresponding wave operators, can clearly be explained as the total energy of neutrinos accompanying the electron-positron system [Sz8]. (This energy term is neglected in the original Ginsburg-Landau-Zeeman Hamiltonian.) Thus these mathematical models really represent systems of charged particles and antiparticles accompanied with electron-positron-neutrinos known in elementary particle physics [V, W1, W2]. Let it be mentioned yet that operator $D_\mu\bullet$ is associated with the magnetic dipole resp. angular momentum operators of classical quantum theory. They are defined for the lattice points separately. Thus, while wandering on the lattice points, these point-like particles receive their angular momenta at the lattice points where they stay on.

Finally, let the isospectrality question be clarified. An isospectrality family is defined by Heisenberg type groups $H_l^{(a,b)}$ having the same l and $(a + b)$, that is, they share the same X-space $\mathcal{X} = \mathcal{Y}^a + \mathcal{Y}^b$ and Z-space $\mathcal{Z}$. They differ from each other just by the decomposition of the X-space and the action of J_Z on these components. When $H_l^{(a+b,0)}$ is compared with $H_l^{(a,b)}$, then the exchange endomorphism σ described above is defined by $id_{/\mathcal{Y}^a}$ resp. $-id_{/\mathcal{Y}^b}$ on the components of the above decomposition. On $\mathcal{X}(b) = \mathcal{Y}^b$, this exchange endomorphism defines the angular momentum of $H_l^{(a,b)}$ by the negative of the angular momentum defined for $H_l^{(a+b,0)}$. One can interpret this as switching the sign of charge of the particles on this component. Now consider an X-ball around the origin and restrict the torus bundle onto this ball. For both bundles, the normal vector at a boundary point is the the radial unit X-vector. For both metrics consider the same complex basis $\mathbf{B}$ and the same polynomials in terms of the complex structures defined for these two metrics. Since both define the same radial Laplacian with the same boundary conditions, these two metrics on the considered sub-bundles must be isospectral regarding any of the boundary conditions. This isospectrality can be seen also by observing that, for any

fixed Z_γ, there exist an orthogonal transformation on the X-space which conjugates $J^{(a+b,0)}_{Z_\gamma}$ to $J^{(a,b)}_{Z_\gamma}$, therefore, it intertwines the Laplacians of the two metrics term by term along with the boundary conditions.

If basis $\mathbf{B}_\gamma$ is chosen by picking the vectors always from subspaces $\mathcal{Y} = \mathbb{R}^{n(l)}$, this isospectrality is a clear manifestation of the spectral C-symmetry. For a general basis, however, it exhibits also internal symmetries. In elementary particle physics, this name was chosen to indicate that one think about internal symmetries as having to do with the intrinsic nature of the particles, rather than their position or motion. You can think of each particle as carrying a little dial, with pointer that points in directions marked "electron" or "neutrino" or "photon" or "W" or anywhere in between. The internal symmetry says that the laws of nature take the same form if we rotate the markings on these dials in certain ways. If a basis vector $Q_i = Q^{(a)}_i + Q^{(b)}_i$ is lying neither in $\mathcal{Y}^{(a)}$ nor in $\mathcal{Y}^{(a)}$, the exchange endomorphism effects only $Q^{(b)}_i$, that is, the particle antiparticle exchange is just partial and not complete. By this explanation, this isospectrality is a clear manifestation of the spectral C/I-symmetry.

7.3 Extended particles occupying Z-ball bundles.

Contrary to the above Z-crystal cases, the Laplacian on Z-ball and Z-sphere bundles can be identified with the Hamilton operators of particles to which interior can be attributed. In elementary particle physics such particles occur in the nucleus where the constituents are held together by the electroweak and strong forces. These forces are exhibited by the idiosyncratic appearance of the Laplacian on these bundles. Namely, it decomposes into a scalar operator and a non-standard spin operator (called also *roulette operator*) where the scalar operator represents the electroweak force while the roulette operator corresponds to the strong force.

The ball×ball- and ball×sphere-type manifolds used to these investigations emerged first in the spectral constructions performed in [Sz4]-[Sz6]. These manifolds are defined by appropriate smooth fields of Z-balls resp. Z-spheres of radius $R_Z(|X|)$ over the points of a fixed X-ball $B_\mathcal{X}$ of radius $R_\mathcal{X}$. Note that radius $R_Z(|X|)$ depends just on the length, $|X|$, of vectors $X \in B_\mathcal{X}$ over which the Z-balls resp. Z-spheres are considered. The boundaries of these manifolds are the so called sphere×ball- resp. sphere×sphere-type manifolds. Comparing with the Z-crystals, the difference between the two type of bundles is that one considers Z-balls resp. Z-spheres instead of the Z-tori used in the previous construction. In the isospectrality investigations

the compact domains corresponding to $R_{\mathcal{X}} < \infty$ play the primary interest. In physics, however, the non-compact bundles corresponding to $R_{\mathcal{X}} = \infty$ (that is, which are defined over the whole X-space) become the most important cases. In the following considerations both cases will be investigated. The details will be provided in this paper just for Z-ball bundles and not for Z-spheres bundles.

The main difference between the Z-crystals and Z-ball bundles is that the computations in the latter case can not be reduced to a single endomorphism but they must be established for the complete operator $\mathbf{M} = \sum \partial_\alpha D_\alpha \bullet$. This operator includes the angular momentum endomorphisms J_Z with respect to any Z-direction. This complication gives rise to a much more complex mathematical and physical situation where both the exterior and the interior life of particle systems exhibit them-self on a full scale.

The above argument implies that this mathematical situation can not be described by the discrete Z-Fourier transform applied on Z-crystals. In this case one considers a fixed complex basis $\mathbf{B}$ together with the complex coordinate systems $\{z_i(K_u, X)\}$ defined, on the X-space, by the complex structures J_{K_u}. Then, the Z-Fourier transform is defined on the whole center, $\mathcal{Z} = \mathbb{R}^l$, by

$$\int_{\mathbb{R}^l} A(|X|, K)\Pi_X^{(n)}(\prod z_i^{p_i}(K_u, X)\overline{z}_i^{q_i}(K_u, X))e^{\mathbf{i}\langle K,Z\rangle}dK. \tag{60}$$

It should be pointed out that a fixed $\mathbf{B}$ can serve as a complex basis only for almost every K_u, which form an everywhere dense open subset on the unit Z-sphere. However, the polynomials are well defined analytic functions even at those K_u's with respect to which the $\mathbf{B}$ is not a complex basis. This formula is well defined if for any fixed $|X|$ function $A(|X|, K)$ is of class L^2 regarding the K-variable.

This so called twisted Z-Fourier transform does not separate but rather binds the Z-space and the X-space together. This strong bond is established by the polynomials $\prod z_i^{p_i}\overline{z}_i^{q_i}$ which depend both on the X- and K-variables. It is proved in Section 6.2 of [Sz8] that the complex valued functions considered behind the integral sign for all possible powers satisfying $\sum_i (p_i+q_i) = n$ span an everywhere dense subspace, $\mathbb{T}w_{\mathbf{B}}^{(n)}$, of the straightly defined complete function space, $\mathbb{S}t^{(n)}$, which can be introduced by extending basis $\mathbf{B}$ into a basis $\tilde{\mathbf{B}} = \{\tilde{Q}_1, \ldots, \tilde{Q}_k\}$ of the whole X-space and replacing the above complex polynomials by the real polynomials $\prod\langle\tilde{Q}_i, X\rangle^{\alpha_i}$, where $\sum \alpha_i = n$. While $\mathbb{T}w_{\mathbf{B}}^{(n)}$ depends on $\mathbf{B}$, this straightly defined function space is independent from the choice of the basis $\tilde{\mathbf{B}}$. This function space naturally

emerges on the Cartesian product $\mathcal{X} \times \mathcal{Z}$ as Cartesian product of the corresponding function spaces. The same statements are true for the Z-Fourier transforms, $\mathbb{F}_Z(\mathbb{T}w_{\mathbf{B}}^{(n)})$ and $\mathbb{F}_Z(\mathbb{S}t^{(n)})$, of these function spaces. When these function spaces are defined for particular powers p_i and q_i, they are denoted by $\mathbb{F}_Z(\mathbb{T}w_{\mathbf{B}}^{(p_i,q_i)})$ resp. $\mathbb{F}_Z(\mathbb{S}t^{(p_i,q_i)})$.

It is also observed in [Sz8] (cf. Theorem 6.1) that, for a function represented by twisted Z-Fourier transform, $A(|X|, K)$ is a uniquely determined L^2_K-function. Thus, by fixing a basis $\mathbf{B}$ and two H-type groups $H_l^{(a,b)}$ and $H_l^{(a',b')}$ satisfying $(a+b) = (a'+b')$, there exists a well defined one to one correspondence $\kappa_{\mathbf{B}} : \mathbb{F}_Z(\mathbb{T}w_{\mathbf{B}}^{(n)}) \to \mathbb{F}_Z(\mathbb{T}'w_{\mathbf{B}}^{(n)})$ that maps a function expressed by $A(|X|, K)$ and complex structures $J_{K_u}^{(a,b)}$ to functions where just the complex structures are exchanged for $J_{K_u}^{(a',b')}$. Let it be pointed out again that this map operates on functions defined by the twisted Z-Fourier transform and not on the ones standing behind the integral sign. This map has a unique extension defining a bijection between the straightly defined function spaces.

In order to check out if this map is an intertwining operator, let the Laplacian be act on the twisted Z-Fourier transform formula. Like for Z-crystals, operator $\mathbf{M} = \sum \partial_\alpha D_\alpha \bullet$ appears behind the integral sign in the form $\mathbf{i}|K|D_{K_u}\bullet$, which acts only on the polynomial part, resulting $|K|m$, where $m = \sum(-p_i + q_i)$. Term involving Δ_Z appears inside as $4\pi^2|K|^2(1 + \frac{1}{4}|X|^2)$. Finally, the radial Laplacian in Δ_X acts on X-radial functions by radial differentiations and multiplications with radial functions, furthermore, the action of the X-spherical Laplacian is nothing but multiplication with the corresponding eigenvalue. Thus the above operator really intertwines the Laplacians (cf. these details in Section 8 of [Sz8]).

Although the considered functions are everywhere dense in the straightly defined complete function space, the boundary conditions can not be computed by them. The main problem is that, regarding the K-variable, function $A(|X|, K)\Pi_X^{(n)}(\prod z_i^{p_i}(K_u, X)\overline{z}_i^{q_i}(K_u, X))$ is not the multiple of a single s^{th}-order spherical harmonic by a K-radial function even if one uses functions $A(|X|, K) = \phi(|X|, |K|)\varphi(K_u)$, where $\varphi(K_u)$ is a spherical harmonic of degree $f = deg(\varphi)$. This claim follows from the fact that function $\prod z_i^{p_i}(K_u, X)\overline{z}_i^{q_i}(K_u, X)$ is not derived from a homogeneous polynomial but, after performing the indicated powering imposed on functions $z_i(K_u, X) = \langle Q_i + \mathbf{i}J_{K_u}(Q_i), X\rangle$ and its conjugate, it appears as a proper

sum of the form:

$$\prod z_i^{p_i}(K_u,X)\overline{z}_i^{q_i}(K_u,X) = \qquad (61)$$

$$\sum_{(a_1\dots a_n)}\prod_i \langle Q_i,X\rangle^{n_i-a_i}\langle \mathbf{i}J_{K_u}(Q_i),X\rangle^{a_i} = \sum_{a=\sum a_i=0}^{n} R^{(a)}(K_u,X),$$

where $n_i = (p_i + q_i)$, $n = \sum n_i$, and index $a = \sum a_i$ indicates that how many complex structures J_{K_u} appear in the term determined by the exponents $(a_1 \dots a_n)$. Sub-sum $R^{(a)}(K_u, X)$ consists exactly those terms where this number is a. Although these functions are derived from a-homogeneous functions, neither they nor $\varphi(K_u)R^{(a)}(K_u, X)$ are spherical harmonics regarding K_u. One obtains the desired s^{th}-order spherical harmonics by projections $\Pi_X^{(n)}$ and $\Pi_K^{(\mathbf{s})}$, where compound index $\mathbf{s} = (s, f, a)$ indicates the degrees both of the target and original functions. Like the projections introduced regarding the X-variable, also the latter projections are the polynomials of the Laplacian Δ_{K_u}. More precisely, they appear in the form $q_{\mathbf{s}}\Pi_K^{(s)}\Delta_{K_u}^{(f+a-s)/2}$, where the first term is a constant and the last term produces from a $(f+a)^{th}$-order polynomial of the K-variable an s^{th}-order polynomial which is then projected to the s^{th}-order spherical harmonics in the same way how it was defined regarding the X-variable. (These computations are described in Section 6.4 of [Sz8].)

The corresponding formula for constructing functions by which the boundary conditions can be handled is:

$$\int_{\mathbb{R}^l}\sum_{\mathbf{s}}\phi_{\mathbf{s}}(|X|,|K|)\Pi_K^{\mathbf{s}}(\varphi(K_u)\Pi_X^{(n)}(\prod z_i^{p_i}\overline{z}_i^{q_i})(K_u,X))e^{\mathbf{i}\langle K,Z\rangle}dK. \qquad (62)$$

It is also pointed out in [Sz8] that for fixed spherical harmonics $\varphi(K_u)$, non-trivial projections are defined just for those s-values which satisfy the inequality $(f - a) \le s \le (f + a)$ and s has the same parity as $(f - a)$ resp. $(f + a)$. Furthermore, functions defined by different a's are projected into independent subspaces. That is, this formula generically involves $(a + 1)$-tuples, $(\phi_{(f-a)}, \dots, \phi_{(f+a)})$, of functions depending $|X|$ and $|K|$.

The functions constructed in this way form a larger space, $\mathbb{F}_Z(\mathbb{LT}w_{\mathbf{B}}^{(p_i,q_i)})$, then those constructed in (60). Since functions (60) form an everywhere dense subspace in the whole straightly defined space, they are everywhere dense also in space spanned by functions defined in (62). This means that in the newly defined space the functions can just be approximated by the previous functions. However, no natural (obvious) approximation exist what

would make the consideration of the new formula evident. Anyhow, the statement about the density of these function spaces is enough to establish the following statement: *For any two members of an isospectrality family, the intertwining operator defined for the primarily functions extends to the newly defined ones such that it associates functions defined in terms of same (X, Z)-radial functions and the corresponding projected functions.* More precisely, this is the only option for a continuous extension, Indeed, if the operator is defined in this way, then on $\mathbb{F}_Z(\mathbb{T}w_{\mathbf{B}}^{(p_i,q_i)})$ it is the same as the operator defined originally. This argument implies that the operator defined in terms of the newly defined functions is still an operator intertwining the Laplacians.

For investigating the boundary conditions there are two important tools established. One of them is the Hankel transform, which is proved in Section 6.3 of [Sz8] in the following form: *Any s^{th}-order spherical harmonic $\zeta_l^{(s)}(K_u)$ defined on the unit sphere of $\mathbb{R}^l$ defines, by the formula*

$$\mathcal{H}_l^{(s)}(\phi)(|Z|)\zeta_l^{(s)}(Z_u) = \int_{\mathbb{R}^l} \phi(|K|)\zeta_l^{(s)}(K_u)e^{\mathbf{i}\langle Z,K\rangle}dK, \tag{63}$$

a uniquely determined transformation $\mathcal{H}_l^{(s)}(\phi)(|Z|)$ on the L^2-radial functions $\phi(|Z|)$ which depends just on s and l. This statement implies that the newly defined function space is appropriate for constructing the complete space both of the Dirichlet and Z-Neumann functions in terms of the (X, Z)-radial functions. It turns out that these two conditions can be characterized as being scalar, meaning that they are satisfied if and only if functions $\mathcal{H}_l^{(s)}(\phi_{\mathbf{s}})(|X|, |Z|)$ appearing in the above formulas satisfy them individually. Actually, these functions are also explicitly described in [Sz8], establishing this statement a much stronger form. As a result, these functions are intertwined by the above operator, indeed.

The other important tool is the *inner algorithm* (cf. Section 6.7 in [Sz8]) induced by the action of the angular momentum operator or operator $D_Z\bullet$ on the second Z-Fourier transform formula. In this algorithm this action is iterated such that in the k^{th}-step the desired functions are constructed by functions $\phi_{\mathbf{s}}^{(k-1)}$ obtained in the previous step by using Hankel transform, radial differentiation, and certain combinations of them which can be described as averaging by the roulette operator [Sz8]. The point is that this process involves just the radial functions and it ends up either in finite or infinite steps. In the latter case the sought functions are obtained by the limiting $k \to \infty$.

There is far more difficult problem to establish the above statement also

for functions yielding the regular Neumann condition. The source of these difficulties is that, contrary to the Dirichlet and Z-Neumann conditions, this one does not break down to single individual functions, but, it can be expressed in terms of all functions $(\phi_{(f-a)}, \dots, \phi_{(f+a)})$. More precisely, with the help of the inner algorithm, one can construct a compound Neumann operator $\mathbb{N}_s^{(f,a)}(\phi_{(f-a)}, \dots, \phi_{(f+a)})$ acting on radial functions such that a function constructed by (62) satisfies the regular Neumann condition if and only if the radial functions defined by the compound Neumann operator vanishes at the boundary points. That is, also this condition can be expressed in terms of radial functions, therefore it is also intertwined by the above operator.

The complexity of angular momentum operator $\mathbf{M}_Z$ is fascinating. Its action can be described by the Hankel transform and the above mentioned inner algorithm in a more precise way (cf. Sections 6.7 and 6.8 of [Sz8]). These tools reveal that it appears as the sum of an extrinsic orbiting operator, $\mathbf{L}$, operating on the radial functions $\phi_{\mathbf{s}}$ without defining permutations (i. e., averaging by the roulette operator) on them, and an intrinsic spin operator, $\mathbf{S}$, acting on radial functions, but contrary to $\mathbf{L}$, it defines also an irreducible permutation on functions ϕ_s. The actual appearance of this operator on the above functions is:

$$\int_{\mathbb{R}^l} \bigcirc_{\mathbf{s}}(\phi_{(f-a)}, \dots, \phi_{(f+a)})\Pi_K^{(\mathbf{s})}(\varphi\Pi_X^{(n)}(\prod z_i^{p_i}\overline{z}_i^{q_i})(K_u, X))e^{\mathbf{i}\langle K,Z\rangle}dK. \quad (64)$$

The intrinsic life of particles is encoded into $\mathbf{S}$. Also the strong nuclear forces, keeping the particles having interior together, can be explained by this operator.

The rest part, $\text{Œ} = \Delta_X + (1 + \frac{1}{4}|X|^2)\Delta_Z + \mathbf{L}$, of the complete Laplacian Δ exhibits just orbiting spin. Its action can be reduced to a single radial function. More precisely, the exterior operator Œ, on constant radius Z-ball bundles reduces to a radial operator appearing in terms of the Dirichlet-, Neumann-, resp. mixed-condition-eigenvalues $\lambda_i^{(s)}$ of the Z-ball $B_Z(R)$ in the form

$$(\diamond_{\mu,\tilde{t}}\mathtt{f})(\tilde{t}) = 4\tilde{t}\mathtt{f}''(\tilde{t}) + (2k + 4n)\mathtt{f}'(\tilde{t}) - (2m\mu + 4\mu^2(1 + \frac{1}{4}\tilde{t}))\mathtt{f}(\tilde{t}). \quad (65)$$

By the substitution $\mu = \sqrt{\lambda_i^{(s)}/4}$, this is exactly the radial Ginsburg-Landau-Zeeman operator (58) obtained on Z-crystal models.

The physical forces corresponding to operator Œ are the weak nuclear forces by which the beta decays are explained. The theoretical establishment of this force went through enormous developments [V, W1, W2]. It

started out with Fermi's theory which was highly surpassed by the Glashow-Weinberg-Salam theory, whose greatest achievement was the unification of the weak nuclear force with the electromagnetic force. The above statement, which is unknown both in physics and mathematics, is an exact mathematical establishment of this unification. In physics the unification with the other forces, that is with strong nuclear forces and gravitation, is one of the most intensely investigated unsolved questions. Since the Z-crystal and Z-ball-bundle models unify the electromagnetic and weak nuclear forces also with the strong nuclear forces, this unification is actually much stronger than those established in the Glashow-Weinberg-Salam theory. The main unifying idea is that all these forces can be derived from the very same operator Δ. They are distinguished only by the invariant subspaces to which the Δ is restricted to. That is, the individual forces emerge on these individual invariant subspaces separately.

Conclusion: *On these mathematical-physical models the investigated isospectralities are equivalent to the spectral C/I-symmetry known in elementary particle physics. If the elements of the basis* $\mathbf{B}$ *are picked up from the invariant subspaces* $\mathcal{X}^{(a)}$ *and* $\mathcal{X}^{(b)}$*, then the isospectrality is the manifestation of the pure spectral C-symmetry. In other words, the isospectrality proofs in my constructions mathematically demonstrate that, regarding the spectrum of the Hamilton operator, these physical models obey the physical C-symmetry law. That is, this spectrum is not changing if some of the particles are exchanged for their antiparticles.*

7.4 Wave mechanics.

The wave operators corresponding to the above Hamilton operators emerge in the Laplacian of the static resp. solvable extensions of the nilpotent groups. In physics, where this operator should be a hyperbolic wave operator, the extended metric must be indefinite having Lorenz signature, where the time axis has signature -1. Regarding both bundles, the solutions of the wave equations considered on the whole bundle can be written up by explicit integral formulas (cf. Section 7 of [Sz8]). On ball-bundles, corresponding to the static and solvable extensions, these de Broglie wave packets appear in the form

$$\int_{\mathbb{R}^l} \phi_s \Pi_K^{(s)} (\varphi \Pi_X^{(n)} (\prod z_i^{p_i} \overline{z}_i^{q_i})) e^{\mathbf{i}(\langle K,Z\rangle - \omega t)} dK, \quad \text{resp.} \tag{66}$$

$$\int_{\mathbb{R}^l} \phi_s \Pi_K^{(s)} (\varphi \Pi_X^{(n)} (\prod z_i^{p_i} \overline{z}_i^{q_i})) e^{\mathbf{i}(\langle K,Z\rangle - \omega e^T)} dK,$$

where ω is a constant depending on the particle system. On Z-crystals, an appropriate discrete version of the integral defines these wave packets. Then, on the extended Z-crystals, the Laplacian is the sum of a Schrödinger operator determined for electron-positron systems and an electron-positron-neutrino operator accompanying the electron-positron system. On ball-bundles, the corresponding particles regarding the Œ-operator are the W- and Z-particles, introduced by Weinberg in his weak-force theory. Analogous particles can be introduced regarding the complete Laplacian $\text{Œ} + \mathbf{S}$. These details are completely described in Section 7 of [Sz8].

On manifolds endowed with indefinite metrics the isospectrality questions can not be raised in general. However, this question can be raised regarding positive definite extensions, where one extends the natural positive definite inner product defined at $(0, 0, 1)$ on the tangent space. These manifolds can be considered as non-relativistic Newtonian space-time models. Actually the harmonic isospectral manifolds discussed in the AriasMarco-Schüth paper appear exactly among these manifolds. The metrics on nilpotent groups are neither harmonic nor Einstein thus they could not have been effected by the AM&S-theorem even if it was a right statement. The proof of isospectrality on the positive definite solvable extensions can be established similarly as on the nilpotent groups. In this case the radial functions should be considered in the form $\phi_s(|X|, |K|, t)$, where $t > 0$ is the new parameter added to (X, Z) which indicates that the solvable extensions are defined by the half-space extension of two-step nilpotent Lie groups. These details and the isospectralities on the boundaries of Z-ball-bundles are described in [Sz6].

The problems arising in my papers [Sz4, Sz5]. The intertwining operators are not correctly established in my papers cited above. These difficulties concern only these papers, where the isospectrality examples are constructed on Z-ball resp. Z-sphere bundles. They do not effect my other constructions performed on Z-torus bundles, and, even in these papers, they concern only the construction of the intertwining operators, while the other statements are unaffected.

These operators are constructed in [Sz4, Sz5] by means of a fixed complex structure $J_0 \in E_{skew}(\mathcal{X})$. The complex structure obtained by σ deformation is denoted by J_0'. Then, the intertwining operator, κ, is defined in these papers by the correspondence

$$\kappa: \quad A(|X|,Z)\Pi_X^{(n)}(\prod_i \langle Q_i + \mathbf{i}J_0(Q_i), X\rangle^{p_i}\langle Q_i - \mathbf{i}J_0(Q_i), X\rangle^{q_i}) \qquad (67)$$

$$\to \quad A(|X|,Z)\Pi_X^{(n)}(\prod_i \langle Q_i + \mathbf{i}J_0'(Q_i), X\rangle^{p_i}\langle Q_i - \mathbf{i}J_0'(Q_i), X\rangle^{q_i})$$

where $Q_i \in \mathcal{X}$ are arbitrary X-vectors. That is, the associated functions appear in terms of the associated complex structures J_0 resp. J_0' in the very same form. It obviously intertwines these two complex structures

It was H. Fürstenau [F] who recognized that this map was not well defined. For if one considers a fixed complex basis $\mathbf{B} = \{B_1, \ldots, B_{k/2}\}$ on the X-space, then the above correspondence can be prescribed only for functions defined by such Q_i's which are in the real subspace $Span_{\mathbb{R}}(\mathbf{B})$ spanned by real linear combinations of the basis-elements B_i. It is evident, that these correspondences will define the action of κ on the rest of functions defined by other Q_i's not lying in $Span_{\mathbb{R}}(\mathbf{B})$. The very same map can be defined in terms of the complex valued polynomials constructed above regarding the basis $\mathbf{B}$. That is, it is enough to consider only the correspondence which associate polynomials (appearing in the same form regarding the associated complex structures) to each other. It is also clear that the action of this well defined κ on functions defined by Q_i's not lying in $Span_{\mathbb{R}}(\mathbf{B})$ differs from the action described in (67). Thus the above map is not well defined, indeed. (Fürstenau used different arguments for explaining this problem.)

This recognition triggered the reconstruction of the above ill-defined intertwining operator. Further investigations showed that a correct operator could have not been established by a single complex structure J_0. Instead, all complex structures J_{Z_u} must be involved to its definition. Even the exchange of J_0 for J_{Z_u} does not alter this newly defined operator into a correct one. It becomes the desired intertwining operator, however, if one applies also the Z-Fourier transform to the latter altered functions. This step is really necessary because the operator defined in terms of functions standing behind the integral sign intertwines just those physical operators which appear after the Laplacian enters behind the integral sign. In terms of a complex basis $\mathbf{B}$ and polynomials written up regarding this basis, the reconstructed intertwining operator can be defined by the functions introduced by the twisted Z-Fourier transforms (60) and (62).

These reconstructed formulas were described first in [Sz6] and lectured about them also at the international conference held at CUNY, in February of 2006 [Sz7]. The physical content reviewed in this paper is the result of a recent development. It is described in [Sz8]. In this paper a completely

new mathematical idea, namely the Hankel transform, is introduced, by which both the electroweak and strong interaction Hamilton operators can be established. By this transform simple transparent proofs can be given also for mathematical theorems such as the density theorems and several other statements concerning intertwining of Laplacians or boundary conditions. In [Sz6] these statements are established by a different integral transform, called dual Radon transform. An other statement used there is the so called independence theorem. It should be pointed out that the exploration of the physical contents inherent in these structures is not quite much efficient by these tools than by the Hankel transform. This statement is certainly true for electroweak and strong interactions, which become understandable just by the Hankel transform. The strength of the latter method can be demonstrated also by the fact that all theorems established in [Sz6] can be established also by this tool.

References

[AM-S] T. Arias-Marco and D. Schüth: Local symmetry of harmonic spaces as determined by the spectra of small geodesic spheres. arXiv:1001.1611. Presented also at the *International Conference on Spectral Geometry*, July 19-23, 2010, Dartmouth College.

[Be] A. L. Besse: Manifolds all of whose geodesics are closed. *Ergebnisse der Math. und Grenz. 93*, Springer Verlag, 1978.

[BCG] G. Besson, G. Courtois, S. Gallot: Entropies et regidités des espaces localement symétriques de courbure strictement négative. *Geom. Func. Anal.*5: 731–799, 1995.

[Bo] D. Bohm: Quantum Theory. *Dover Publications, INC., New York*, 1979

[DR] E. Damek and F. Ricci: A class of non-symmetric harmonic Riemannian spaces. *Bull. Amer. Math. Soc. (N.S.)*:27; no. 1, 139–142, 1992.

[F] H. Fürstenau: Über Isospektralität von topologischen Bällen. *Diploma Thesis* Univ. Bonn, 2006.

[GW] C. Gordon and E. Wilson: Continuous families of isospectral Riemannian metrics which are not locally isometric. *J. Diff. Geom.*47: 504–529, 1997.

[H] J. Heber: On harmonic and asymptotically harmonic homogeneous spaces. *Geom. Funct. Anal.*16: 869–890, 2006.

[K] A. Kaplan: Riemannian nilmanifolds attached to Clifford modules. *Geom. Dedicate*, 11:127–136, 1981.

[Kn] G. Knieper: New results on noncompact harmonic manifolds . *arXiv:0910.3872.*

[LL] L. D. Landau, E. M. Lifshitz: Quantum Mechanics. Pergamon, 1958.

[N] Y. Nikolayevsky: Two theorems on harmonic manifolds. *Comm. Math. Helv.*80: 29–50, 2005.

[P] W. Pauli: Pauli Lectures on Physics. Dover, 2000.

[RS] A. Ranjan and H. Shah: Harmonic manifolds with minimal horospheres. *J. Geom. Anal.*12: 683–694, 2002.

[S] D. Schüth: Isospectral manifolds with different local geometries. *J. Reine Angew. Math.* 534: 41–94, 2001.

[Sz1] Z. I. Szabó: Lichnerowicz conjecture on harmonic manifolds, *J. Diff. Geom.* 31: 1–28, 1990.

[Sz2] Z. I. Szabó: Spectral theory for operator families on Riemannian manifolds. *Proc. of Symp. in Pure Math.*, 54: 615–665, 1993.

[Sz3] Z. I. Szabó: Locally non-isometric yet super isospectral spaces. *Geom. funct. anal. (GAFA)*, 9:185–214, 1999.

[Sz4] Z. I. Szabó: Isospectral pairs of metrics on balls, spheres, and other manifolds with different local geometries. *Ann. of Math.*, 154:437–475, 2001.

[Sz5] Z. I. Szabó: A cornucopia of isospectral pairs of metrics on spheres with different local geometries. *Ann. of Math.*, 161:343–395, 2005.

[Sz6] Z. I. Szabó: Reconstructing the intertwining operator and new striking examples added to "Isospectral pairs of metrics on balls and spheres with different local geometries". *DG/0510202.*

[Sz7] Z. I. Szabó: Laplace spectra on open and closed Zeeman manifolds. *math.SP/0602474.*

[Sz8] Z. I. Szabó: Exact mathematical models of a unified quantum theory; Expanding and static micro universes. *arXiv:0909.3902.*

[V] M. Veltman: Facts and Mysteries in Elementary Particle Physics. *World Scientific,* 2003.

[W1] S. Weinberg: Dreams of a Final Theory. *Vintage Books*, A Division of Random House, Inc., 1994.

[W2] S. Weinberg: The Quantum Theory of Fields I;II. *Cambridge Univ. Press*, 1995.